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WAVEGUIDES FILLED WITH MAGNETOPLASMAS
OF VARIOUS TYPES

by

Inci Akkaya

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Wave propagation in a linearized, homogeneous magnetoplasma of bounded and unbounded regions has been investigated in detail. Plasmas of various models are considered in order to evaluate the effects of electron temperature and of the resonances on the waves. In this thesis three models are studied: the usual incompressible model, the compressible fluid model and a microscopic model. The second model is based on the transport equations and the third on the Boltzmann equation with an assumed Maxwellian velocity distribution at equilibrium.)

Waves in a circular waveguide filled with either incompressible or compressible plasma are analyzed under the boundary condition that the tangential electric field is zero and for the latter case the normal electron velocity is zero on the guide wall. When the static magnetizing field is parallel to the guide axis, the modal waves can be expressed in terms of known functions. The longitudinal propagation constants for both models are numerically evaluated and compared. In the case of the compressible plasma, the electroacoustic waves are coupled to the optical waves through the magnetizing field and the boundary conditions. As a consequence the propagation constants are found to consist of two types. The first can be identified as those of the incompressible model, but slightly perturbed. Modes of the second type arise from the compressibility of the plasma and the assumed boundary conditions for the electron velocity. For the second

type, since the acoustic speed is much smaller than the light speed, the permissible propagation constants are so densely located in the Brillouin diagram that they resemble a continuous spectrum.

Modal fields due to a source in the waveguide filled with compressible or incompressible plasmas are formulated. The relationship between the power and the impedance of an antenna in the guide is determined. A numerical example is given for the resistance of a small antenna in the guide filled with incompressible plasma.

Finally, the case of oblique static magnetizing field to the guide axis is discussed. Unfortunately, the modal fields for this case cannot be represented by known functions.

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I. INTRODUCTION

Ionized medium in the presence of a dc magnetic field, namely, magnetoplasma has been subject of intense interest in recent years. Most investigations have been devoted to wave propagation in plasma of unbounded regions. Since the laboratory produced plasmas are confined to small finite regions only, often comparable to the free space wavelength of interest, it appears to be appropriate to investigate waves in a waveguide filled with anisotropic plasma. But the complication in such a problem arises from the requirement of satisfaction of certain boundary conditions.

When a wave propagates in an anisotropic plasma medium, very complex physical processes take place. In order to gain some theoretical understanding of the phenomena, as usual, various idealizations are imposed on the system and thus result in different plasma models. These idealizations are used to simplify some aspects of the medium, such as the governing equations, convection currents and interactions of different species of ions, the Brownian motion of charged particles, the homogeneity of the medium, the boundary conditions on the conducting walls, etc.

The simplest mathematical description of the waves in plasma consists of the Maxwell equations, the continuity equation of the mass flux, and the equation of motion for a single particle. Even for such a simple model, the solution is exceedingly difficult to obtain due to the nonlinearity of the equation. It is usually assumed that the medium is at equilibrium and the RF signal is so weak that the perturbations

of the medium are small enough to validate a linearized theory. Moreover often only the ion convection current which is due to electrons is considered, because of the high mobility of the electrons as compared with other ions. Collisions among charged particles are also very complicated phenomena; however, since collisions among particles of the same species result in no net change of momentum, sometimes their effect is neglected altogether, particularly when the wave frequency is high. Other times a simple correction may be obtained by replacing the electron mass m_e by a fictitious mass $m_e(1-j\nu/\omega)$ where ν is the so-called collision frequency.

For a low temperature plasma the thermal motion of the charged particles may be ignored. This results in the well known cold plasma model⁽¹⁾ which has been successfully used to explain many ionospheric phenomena. However, near resonance the wave velocity becomes close to zero, so that neglecting the thermal motion ceases to be a good approximation. To take into account the thermal motion, the plasma is generally treated as a compressible fluid. A new quantity, namely the pressure must be used to describe the magnetoplasma somewhat more precisely. For this new unknown it is necessary to add an equation, usually the equation of state, to those already stated. This constitutes the so-called transport theory of a warm or compressible plasma model. From a more precise microscopic description of the plasma, namely the Boltzmann equation, it is known that even for such a model, the low temperature approximation has been implicitly used. Nevertheless, because of the introduction of compressibility, new phenomena, such as the electroacoustic waves, may be explained.

In the warm plasma model, phenomena which are associated with the velocity distribution of charged particles are not included. For a more precise description the Boltzmann equation may be employed. The first two moments of this equation over the velocity space reduce to the equations of conservation of particles and momentum or the continuity equation and the equation of motion. The third moment equation relates the pressure tensor and the heat flow triadic.⁽²⁾ By neglecting the latter and assuming the pressure to be a scalar which is related to the particle density through the equation of state, one obtains exactly the transport theory of a warm plasma model. The Boltzmann equation is an equation for the one particle distribution function, which is difficult to solve, because of, among other difficulties, the lack of knowledge of the collision integral term. By assuming that the collision integral is simply proportional to perturbed velocity distribution function (although this assumption is not compatible with conservation of the number of particles) assuming some linearization to the equation, Allis, Buchsbaum and Bers⁽³⁾ were able to obtain an approximate solution to the dispersion relation. However, their results are limited to the case of very high magnetizing field and low temperature and also not valid for arbitrarily large wave number.

In this thesis the aforementioned three models will be considered and later applied to a plasma filled circular waveguide. However, because of the complexity of the third model, most of the numerical results are evaluated for the first two models only. The

necessity of considering the warm plasma model arises from the fact that at resonance the electron thermal velocity can no longer be ignored as stated earlier.

Waves guided in parallel planes, rectangular and circular pipes filled with magnetoplasmas or ferrites have been investigated by many authors. Unz⁽⁴⁾ studied the parallel plates filled with a ferrite magnetized in an arbitrary direction and obtained a characteristic equation for the propagation constant which must be solved with a computer. Kales, Cherit and Sakiotis⁽⁵⁾ and later Brazilai and Gerosa⁽⁶⁾ investigated the anisotropic rectangular waveguide. In this case only when the dc magnetizing field \underline{B}_0 is perpendicular to the guide walls, and all fields are independent of the coordinate along \underline{B}_0 , the E- and H-waves become uncoupled, and thus the problem can be easily solved. The dual to this problem, namely when the plasma is replaced by ferrite was considered by Kales. He also studied the case when \underline{B}_0 was parallel to the guide axis⁽⁷⁾ and found that there exists no uncoupled TE or TM waves except at cut-off. For the latter the cut-off frequency can be easily determined. Later Epstein⁽⁸⁾ considered the circular guide and obtained modal solutions in terms of known functions, if \underline{B}_0 is parallel to the guide axis; however, he gave no numerical results.

For anisotropic circular waveguides with axial dc magnetization, a small number of numerical solutions for the cut-off frequencies and the propagation constants have been obtained by various authors.

Two of these works give significant results. Suhl and Walker⁽⁹⁾ follow a very involved approach. Their report which is mostly concerned with ferrite filled guides gives several diagrams for cut-off frequencies as well as for propagation constants. They use two parameters one of which would correspond to the inverse of Y for an anisotropic plasma. Naming the modes according to their limiting forms when B_0 become zero, they obtain results for TE_{11} , TE_{12} and TM_{11} modes. They work on magnetoplasma filled guides also, and give one curve for the cut-offs of the TE_{11} mode. A systematic work considering cold plasma is given by Bevc and Everhort.⁽¹⁰⁾ They name the modes according to their cut-off forms. Their report includes several diagrams of cut-off frequencies for modes with the first three indices of angular dependence and the first three solutions for each angular index. Also Brillouin diagrams for TM_{01} , TM_{02} , TM_{11} , TE_{01} and TE_{11} modes are given.

In this paper on axially magnetized plasma filled waveguides, Willet⁽¹¹⁾ considers the effect of a pressure gradient. He assumes that on the conducting boundary the tangential electric field, the normal RF magnetic field and the normal convection current components are zero. Actually his assumption for the magnetic field is the direct result of the assumed condition on the electric field. In his momentum equation the electric field does not play any role. In his so-called generalized Ohm's law, on the other hand, the net effect of B_0 comes out to be zero. He adds a resistivity term which should correspond to collisions. After

dropping some terms he obtains his approximate equations and finds TE and TM modes for guide of arbitrary cross section. Then he gives examples, namely for rectangular and circular cross sections.

Several studies have been made regarding the power and orthogonality properties of modal waves in anisotropic media. Buchsbaum⁽³⁾ analyses the power carried in anisotropic guides for two models, namely the cold and the warm plasmas. Collin in his book⁽¹²⁾ indicates that if an anisotropic guide shows reflection symmetry, simple orthogonality relations exist among modes.

The purpose of this work is first to reformulate the problem of anisotropic waveguide in a unified but simpler and more straightforward approach than those taken by other authors; second to study the waves in a warm anisotropic plasma; third to determine the wave propagation characteristics in a circular waveguide filled with plasma and fourth to determine the effect of plasma temperature on propagation characteristics. We also consider an anisotropic waveguide with an oblique magnetization in order to determine whether modal solutions in terms of known functions can be obtained.

In this thesis a general expression for dispersion relation of an anisotropic plasma is derived and then applied to three different plasma models, namely the incompressible, the compressible fluid model and the one obtained by using Boltzmann's theory. In all these cases we have made use of the assumptions discussed at the very beginning. It is found that the compressible fluid model brings forth some modifica-

tions to the cold plasma index surfaces. Also the results obtained from the Boltzmann's theory showed that, the compressible fluid plasma refractive indices need some modifications for frequencies very close to the resonances of the electrons.

For the waveguide the boundary condition is assumed to be that the tangential electric field on the conducting surface vanishes for all models. In the case of warm plasma model, it is further assumed that the normal component of the convection current vanishes on the conducting surface.

The power, impedance and orthogonality relations of waves in the anisotropic guide are studied in rather general terms for both incompressible and compressible models.

For circular waveguides with axial magnetization, it is found that simple orthogonality relations can be derived for the warm plasma as well as for the cold plasma.

As for the propagation constants of the guide it is interesting to find that they can be divided mainly into two types. First, there are some solutions of the characteristic equation of the warm plasma model, which differ little from those for the cold plasma model; in fact, they reduce to the latter as the plasma temperature approaches zero. To the second category belong solutions which are strongly dependent on the guide parameter and may vary widely for a small change in frequency.

Fields associated with the modes of the aforementioned three types of propagation indices have been studied. It is found that fields

of the first type have a form somewhat modified from that of the corresponding fields in the cold plasma model. The only significant difference appears in the radial electric and azimuthal magnetic field components. The fields of the second type have their main contribution from the plasma waves and behave like a TEM mode, although it is a combination of three fields each of which has different transverse propagation constants. These fields correspond to the ordinary, extraordinary and plasma waves. For all the three cases, the convection current is axial, and the RF plasma density as well as the convection current are mainly supplied by the plasma wave.

Since in axially magnetized waveguides only the fields with real and purely imaginary axial propagation constant $j\gamma$ contribute to the real and reactive power respectively, the subsequent investigation on the roots of the characteristic equations is confined solely to these two kinds. To facilitate the computation it is further restricted to the solutions of the case where the characteristic equation is real. It is found that along the real axis of γ^2 , only certain regions satisfy this condition. For a cold plasma, with given values of X and Y , the number of these regions can be at most two. For warm plasmas, with fixed temperature, X and Y , the maximum number of these regions is three. The lower and upper bounds of these regions do not depend upon the normalized radius $2\pi r_0/\lambda$ nor the azimuthal behavior of the fields but they depend upon X , Y and temperature only. For cold plasma, with $0 < X \leq 3$ and $0 < Y \leq 2$, these bounds are computed. For both cold and warm models with fixed plasma density and B_0 , the axial refractive index

is numerically determined as a function of the normalized radius, or the frequency for a given radius. The results obtained using the exact form of the characteristic equation of the warm plasma are found to be in perfect accordance with those which the study of an approximated characteristic equation predicts. General expressions, for the input impedance of an antenna placed in the guide filled with either a cold or a warm plasma, are derived in terms of assumed distribution of the current and the pressure. For an example the input resistance of a probe is numerically determined using the cold model.

For B_0 oblique to the guide axis the field equations are studied in the case of cold plasma. It is found that the solutions can be expressed in terms of infinite series and the determination of the coefficients of these series is very complicated, even for the case of uniaxial plasma.

The first chapter of this thesis deals with the general dispersion relation which is applicable to the three aforementioned plasma models. A short comparative study of these three cases is also given. In Chapter II, field expressions of both incompressible and compressible fluid models are given in general terms. The boundary conditions of the waveguide are imposed and the characteristic equations for the case of axial magnetization are obtained. These equations, as well as the fields corresponding to their solutions are compared in order to determine the effect of plasma temperature. Chapter III is

concerned with the orthogonality and power relations of a warm plasma, including the cold plasma as a special case. In Chapter V, numerical results are given for cold and warm plasmas. The derivation of the impedance of an antenna in the waveguide is studied in Chapter VI for the warm plasma, again including the cold plasma model as a special case. Using the cold model, the radiation resistance of a probe is computed. In Chapter VII, a waveguide with a B_0 oblique to its axis is studied.

The study of the dispersion surfaces, the field expressions, characteristic equation, orthogonality and impedance relations for the warm plasma model and their comparison with those for the cold plasma can be summarized as follows:

Part of the solutions for the warm plasma model are slight modifications of those for the cold plasma model; whereas the remaining ones are essentially attributed to the plasma pressure waves. At places where two of the three characteristic waves have their wave number or their wave number and field strength of the same order of magnitude, hybrid solutions take place. Infinities or zeros of the cold plasma results, however, are considerably modified by the temperature effect. For example, the infinities of cold plasma index surfaces become finite and, instead, at that angle there are a real and an imaginary propagation index with almost equal magnitude; as another example, the so-called TE cutoff wave of the cold plasma mode is replaced with a TM warm plasma mode.

Although the compressible fluid model is an improved description over the cold plasma model, other phenomena which are associated with the velocity distribution of electrons are still ignored. In the first Chapter, by using the Boltzmann equation approach it is found that the propagation transverse to B_0 may yield arbitrarily large refractive index near gyroresonance or its multiples. A more precise and complete study of the Boltzmann equation is rather complicated and it seems that only a numerical approach would be possible.

II. DISPERSION RELATIONS FOR VARIOUS PLASMA MODELS

2.1 Field Equations for a General Case

Let us assume that we have a medium filled with an anisotropic dielectric with a relative permeability dyadic $\underline{\underline{K}}$ or matrix (K)

$$(K) = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix} \quad (2-1)$$

Let us further assume that we have*

$$K_{xy} = -K_{yx}; \quad (2-2)$$

$$K_{xz} = K_{zx}, \quad (2-3)$$

$$K_{zy} = -K_{yz} \quad (2-4)$$

If the medium is lossless, the matrix (K) becomes Hermitian (Appendix 1).

The Maxwell Equations for this medium are:**

*In our study we will consider only the media with the property given by Equations (2-2) through (2-4).

**In this study the following notations will be used to denote partial derivatives:

$$\begin{aligned} d_t &\rightarrow \partial/\partial t \\ d_x &\rightarrow \partial/\partial x \\ d_y &\rightarrow \partial/\partial y \\ &\dots \dots \dots \\ d_{xy} &\rightarrow \partial^2/\partial x \partial y \\ d_{xx} &\rightarrow \partial^2/\partial x^2 \\ &\dots \dots \dots \\ d_v &\rightarrow \partial/\partial v \\ d_w &\rightarrow \partial/\partial w \\ &\dots \dots \dots \end{aligned}$$

$$\text{Curl } \tilde{\mathcal{H}} = d_t \tilde{\mathcal{D}} + \tilde{\mathcal{J}} = d_t \epsilon_0 (\tilde{\mathbf{K}} \cdot \tilde{\mathcal{E}}) + \tilde{\mathcal{J}} \quad (2-5)$$

$$\text{Curl } \tilde{\mathcal{E}} = -d_t \tilde{\mathcal{B}} - \tilde{\mathcal{K}} = -d_t \mu_0 \tilde{\mathcal{B}} - \tilde{\mathcal{K}} \quad (2-6)$$

Solving for $\tilde{\mathcal{E}}$ in absence of sources ($\tilde{\mathcal{J}} = \tilde{\mathcal{K}} = 0$) we get

$$(\text{grad div} - \nabla^2) \tilde{\mathcal{E}} = -d_{tt} \mu_0 \epsilon_0 \tilde{\mathbf{K}} \cdot \tilde{\mathcal{E}} \quad (2-7)$$

Let us apply Fourier transforms to all the quantities with respect to t and z such that the transform will take the time variable t to the angular frequency ω and the spatial variable z to $-j\gamma$ $j\gamma$ being the propagation constant along the z axis. Denote the Fourier transforms of $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{H}}$ as $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ respectively. Then in the transformed domain Equation (7) becomes

$$\tilde{\mathbf{g}} \cdot \tilde{\mathbf{E}} = 0 \quad (2-8)$$

where

$$(\mathbf{g}) = \begin{bmatrix} -d_{yy} - \gamma^2 & d_{xy} & \gamma d_x \\ d_{xy} & -d_{xx} - \gamma^2 & \gamma d_y \\ \gamma d_x & \gamma d_y & -d_{xx} - d_{yy} \end{bmatrix} - (\mathbf{k}) \quad (2-9)$$

where

$$\tilde{\mathbf{k}} = k_o^2 \tilde{\mathbf{K}} \quad (2-10)$$

where

$$k_o^2 = \omega^2 \epsilon_0 \mu_0 \quad (2-11)$$

From the set of Equations (8), (9), and (10) one obtains

$$\Delta E_i = 0 \quad i = x, y \text{ or } z. \quad (2-12)$$

where Δ is the operator which can be represented as the determinant of the coefficient matrix of Eqs. (8), (9), and (10).*

*Since in Eqs. (8), (9), and (10) the operations involve differentiations, multiplications and additions only, and since these operators are commutative, excluding the division, the rules of algebra can be applied to solve the system of linear equations. Then with a little manipulation, one arrives at Eq. (13).

Let the terms of matrix (g) be defined as follows:

$$(g) = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} . \quad (2-13)$$

and the cofactor matrix of (g) as

$$(G) = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \quad (2-14)$$

Using the above definitions one can write the following equations:*

$$A_1 E_y = A_2 E_x, \quad (2-15.1)$$

$$A_1 E_z = A_3 E_x, \quad (2-15.2)$$

$$A_2 E_z = A_3 E_y, \quad (2-15.3)$$

$$B_1 E_y = B_2 E_x, \quad (2-16.1)$$

$$B_1 E_z = B_3 E_x, \quad (2-16.2)$$

$$B_2 E_z = B_3 E_y, \quad (2-16.3)$$

*The indicated results have been derived on the same line of reasoning as explained in the footnote of Eq. (13).

$$C_1 E_y = C_2 E_x, \quad (2-17.1)$$

$$C_1 E_z = C_3 E_x, \quad (2-17.2)$$

$$C_2 E_z = C_3 E_y. \quad (2-17.3)$$

Therefore, the general solution of Eq. (2-7) can be given as:

$$\Delta \pi = 0 \quad (2-18.1)$$

$$\vec{E} = \sum_i \vec{e}_i \cdot \vec{G} \pi \quad (2-18.2)$$

The above defined scalar function π can be called the potential function of the field in the magnetoplasma.

where \vec{e}_i can be chosen as any of the unit base vectors.

Because of their comparatively shorter expressions in this work we will be dealing with C_1 , C_2 and C_3 rather than with the other elements of (G) . Namely, in applying Eq. (2-18.2) we will choose

$$\vec{e}_i = \vec{z}.$$

The expressions for C_i 's are given below:

$$\begin{aligned} C_1 = & \gamma^2 d_x^2 \nabla^2 - k_{yz} d_{xy} - k_{xz} d_{xx} - \gamma k_{xy} d_y + \gamma k_{yy} d_y \\ & + \gamma^3 d_x^3 - (\gamma^2 + k_{yy}) k_{xz} + k_{xy} k_{yz} \end{aligned} \quad (2-18.1)$$

$$\begin{aligned}
C_2 = & \gamma d_y \nabla_t^2 - k_{xz} d_{xy} - k_{yz} d_{yy} + \gamma k_{xy} d_x + \gamma k_{xx} d_y \\
& + \gamma^3 d_y - k_{xz} k_{xy} - (\gamma^2 + k_{xx}) k_{yz}
\end{aligned} \tag{2-18.2}$$

$$\begin{aligned}
C_3 = & \gamma^2 \nabla_t^2 + k_{yy} d_{yy} + k_{xx} d_{xx} + (\gamma^2 + k_{xx})(\gamma^2 + k_{yy}) \\
& + k_{xy}^2
\end{aligned} \tag{2-18.3}$$

where

$$\nabla_t^2 = d_{xx} + d_{yy} \tag{2-19}$$

The expression for Δ can be given as

$$\begin{aligned}
\Delta = & [-2\gamma k_{xz} d_x - k_{xx} k_{yy} - k_{xy}^2 - \gamma^2 k_{zz}] \nabla_t^2 \\
& + k_{xz}^2 d_{xx} - k_{yz}^2 d_{yy} - (k_{xx} d_{xx} + k_{yy} d_{yy})(\gamma^2 + \nabla_t^2 + k_{zz}) \\
& + 2\gamma [-(\gamma^2 + k_{yy}) k_{xz} + k_{xy} k_{yz}] d_x \\
& - \gamma^4 k_{zz} + \gamma^2 [k_{xz}^2 - k_{yz}^2 - (k_{xx} + k_{yy}) k_{zz}] - D
\end{aligned} \tag{2-20}$$

where

$$-D = -2k_{xy} k_{yz} k_{xz} + k_{yy} k_{xz}^2 - k_{xx} k_{yz}^2 - k_{xx} k_{yy} k_{zz} - k_{xy}^2 k_{zz}.$$

Δ includes differentiations of 6th order. Fourier transformed forms of Eqs. (2-5) and (2-6) would yield 6 linear equations in terms of $H_x, H_y, H_z, E_x, E_y, E_z$ and the determinant of the coefficients of these equations would involve derivatives of 6th order. Therefore, this determinant must be equal to Δ . Hence, Eq. (2-12) can be generalized as

$$\Delta F = 0, (F = E_x, E_y, E_z, H_x, H_y \text{ or } H_z). \tag{2-21}$$

2.2 Field Equations for Cold Plasma

For a cold plasma with a uniform dc magnetic field \underline{B}_0 , the convection current carried by the electrons can be eliminated from the Maxwell equations by first solving this current in terms of the electric field, namely the generalized Ohm's law:

$$\underline{J} = \epsilon_0 \underline{X} \cdot \underline{E}$$

and then defining the dyadic \underline{K} of relative permeability as

$$\underline{K} = \underline{I} + \underline{X}$$

where \underline{I} is the idemfactor.

In case one takes the z axis parallel to \underline{B}_0 and considers only the contribution of electrons to the convection current, one finds that the matrix (K) reduces to

$$(K) = \begin{bmatrix} K_{xx} & K_{xy} & 0 \\ -K_{xy} & K_{xx} & 0 \\ 0 & 0 & K_{zz} \end{bmatrix} \quad (2-22.1)$$

where

$$K_{xx} = 1 - [X(1-jZ)] / [(1-jZ)^2 - Y^2] \quad (2-22.2)$$

$$K_{xy} = j XY / [(1-jZ)^2 - Y^2] \quad (2-22.3)$$

$$K_{yy} = K_{xx} \quad (2-22.4)$$

$$K_{zz} = 1 - X / (1-jZ) \quad (2-22.5)$$

where

$$X = Ne^2 / (m\epsilon_0 \omega^2) = \omega_N^2 / \omega^2 \quad (2-22.6)$$

$$Y = eB_0/m\omega = \omega_H/\omega \quad (2-22.7)$$

$$Z = \nu/\omega \quad (2-22.8)$$

N = electron density

$$B_0 = |\vec{B}_0|$$

ν = the average collision frequency of electrons

m = mass of electron

$-e$ = charge of electron

ω_N = angular plasma resonance frequency

ω_H = angular gyroresonance frequency.

In this case the expression for Δ becomes

$$\begin{aligned} \Delta = & -k_{xx} \{ \nabla_t^4 + [\gamma^2 + k_{zz} + k_{xx} + (k_{xy}^2/k_{xx}) + \gamma^2(k_{zz}/k_{xx})] \nabla_t^2 \\ & + \gamma^4(k_{zz}/k_{xx}) + 2\gamma^2 k_{zz} + (D/k_{xx}) \} \end{aligned} \quad (2-23)$$

where k_{xx} , k_{xy} , k_{zz} and D depend on the plasma parameters only.

Using the following definitions:

$$\gamma^2 + k_{zz} + k_{xx} + (k_{xy}^2/k_{xx}) + \gamma^2(k_{zz}/k_{xx}) = 2P \quad (2-24.1)$$

$$\gamma^4(k_{zz}/k_{xx}) + 2\gamma^2 k_{zz} + (D/k_{xx}) = Q \quad (2-24.2)$$

$$\nu_{1,2}^2 = P \pm (P^2 - Q)^{1/2} \quad (2-24.3)$$

This is the dispersion relation for the cold plasma giving the transverse component of the refractive index in terms of the longitudinal component.

Eq. (2-18) assumes the following form*

$$-k_{xx} (\nabla_t^2 + \nu_1^2) (\nabla_t^2 + \nu_2^2) \pi = 0 \quad (2-25)$$

*The dual of this result for a gyromagnetic medium coincides with the one given by Epstein.⁽⁸⁾

Eq. (2-25) implies that either

$$(\nabla_t^2 + v_1^2)\pi = 0. \quad (2-26)$$

or

$$(\nabla_t^2 + v_2^2)\pi = 0. \quad (2-27)$$

If we let the solution of Eq. (2-26) be

$$\pi_1 = f_1(x,y)e^{Yz} \quad (2-28)$$

and the solution of Eq. (2-27) be

$$\pi_2 = f_2(x,y)e^{Yz} \quad (2-29)$$

then because of the linearity of Maxwell equations

$$\pi = [\delta_1 f_1(x,y) + \delta_2 f_2(x,y)] e^{Yz} \quad (2-30)$$

where δ_1 and δ_2 are constants.

For cold plasma the elements of the third row of (G) become:

$$C_1 = Y d_x \nabla_t^2 - Y k_{xy} d_y + Y k_{yy} d_x + Y^3 d_x \quad (2-31.1)$$

$$C_2 = Y d_y \nabla_t^2 + Y k_{xy} d_x + Y k_{xx} d_y + Y^3 d_y \quad (2-31.2)$$

$$C_3 = (Y^2 + k_{xx}) \nabla_t^2 + (Y^2 + k_{xx})^2 + k_{xy}^2 \quad (2-31.3)$$

Since according to Eqs. (2-26) and (2-27) together Eq. (2-18)

$$\nabla_t^2 E_{ij} = -v_j^2 E_{ij}; \quad j = 1,2 \quad (2-32)$$

hence, $C_3 E_{ij}$ has the following somewhat simpler form:

$$C_3 E_{ij} = q E_{ij}; \quad (j = 1,2) \quad (2-33)$$

where q is a constant.

2.3 Field Equations for Warm Plasma Based Upon the Transport Theory

The \underline{K} which will be used in this section is given in terms of the propagation vector \underline{k} . Therefore, the equations to be considered have to be worked out in a Fourier transformed domain, not only for the variables t and z but also for the variable x . In this case without the loss of the generality the x axis is chosen to be parallel to the plane formed by the z axis (which is parallel to \underline{B}_0) and the propagation vector \underline{k} . Therefore, the variable y does not come into the picture. Due to the temperature effect the electrons have a velocity distribution function f which determines the density of the electrons N as a function of their velocity \underline{v} according to the following equation

$$dN = f(\underline{v}) dv_x dv_y dv_z$$

Therefore, the convection current \underline{I} will now be found via the equation

$$\underline{I} = -e \iiint \underline{v} f dv_x dv_y dv_z$$

The equation of electron motion used in obtaining the \underline{K} of the cold plasma has now to be replaced by the Boltzmann equation

$$d_t f + \underline{v} \cdot \underline{\nabla} f - (e/m) (\underline{E} + \underline{v} \times \underline{B}) \cdot (df/d\underline{v}) = (\partial f / \partial t)_{\text{collisions}} \quad (2-34)$$

From the solution of this equation one can determine the conductivity dyadic $\underline{\chi}$ for the Ohm's law:

$$\underline{\underline{I}} = \epsilon_0 \underline{\underline{X}} \cdot \underline{\underline{E}}. \quad (2-35.1)$$

where

$$\underline{\underline{K}} = \underline{\underline{I}} + \underline{\underline{X}}. \quad (2-35.2)$$

There are two methods for the determination of $\underline{\underline{X}}$. In the following we shall consider the first while the second method will be postponed until the next section. In essence the first is to replace the microscopic description of the plasma, which is expressed with the velocity distribution function and the Boltzmann equation, by a macroscopic one. This is usually achieved by taking various moments of the Boltzmann equation over the velocity space. Unfortunately, each time a higher order moment is taken a new macroscopic quantity is introduced and the complexity also increases very rapidly. In practice this method is seldom carried beyond the second moment, which leads to the equation of the conservation of momentum with a newly introduced hydrodynamical quantity, the pressure tensor. For a simple approximate theory, the pressure is assumed to be scalar and related to the density through the equation of state. This constitutes the so-called Boltzmann transport theory.

In following this procedure, one first multiplies the equation by 1 and $\underline{\underline{v}}$ respectively and integrates over the velocity space, then obtains the following transport equations

$$e \, d_t N - \underline{\underline{\nabla}} \cdot \underline{\underline{I}} = 0 \quad (2-36)$$

$$-(m/e) d_t \underline{\underline{I}} + \underline{\underline{\nabla}} \cdot \underline{\underline{P}} + Ne \underline{\underline{E}} - \underline{\underline{I}} \times \underline{\underline{B}}_0 = 0 \quad (2-37)$$

where \underline{P} is defined as

$$\underline{P} = \iiint m \underline{v} \underline{v} f dv_x dv_y dv_z.$$

For simplicity usually \underline{P} is assumed to be a scalar

$$\underline{P} = P \underline{I}$$

where

$$P = N^l \quad (2-38)$$

and

$$l = (\alpha + 2)/\alpha. \quad (2-39)$$

Here α is the number of degrees of freedom of the motion of the electrons.⁽¹³⁾

From the set of the five equations, namely Eqs (2-36), (2-37), and (2-38) together with the Maxwell equations two of which are scalar and three of which are vectorial one can eliminate the quantities N , P and \underline{I} and obtain Maxwell equations with the modified \underline{K} which is defined by Eqs. (2-34) and (2-35). The expression for the matrix (K) is

$$(K) = I - F(n) \cdot \begin{bmatrix} 1 - W n_p^2 & jY(1 - W n_p^2) & W n_p n_t \\ -jY(1 - W n_p^2) & 1 - W n^2 & -jY W n_p n_t \\ W n_p n_t & jY W n_p n_t & 1 - W n_t^2 - Y^2 \end{bmatrix} \quad (2-40.1)$$

where I is the identity matrix,

$$F(n) = X / [1 - Y^2 - W(n^2 - Y^2 n_p^2)] \quad (2-40.2)$$

$$W = l k T \epsilon_0 \mu_0 / m = (a/c)^2 \quad (2-40.3.1)$$

a = acoustic speed in the medium

k = Boltzmann's constant = $(1.3804)10^{-16}$ erg/°K

or

$$W = (1.686)10^{-10} \ell T \quad (2-40.3.2)$$

$$n_p = j\gamma/k_o \quad (2-40.4)$$

$$n_t = k_t/k_o \quad (2-40.5)$$

$$n^2 = n_p^2 + n_t^2 \quad (2-40.6)$$

k_t , in the present case is the propagation constant in the x direction. Thus we assume that all field quantities vary as $\exp[-j(k_t x + k_p z)]$.

Now inserting Eq. (2-40) into Eqs. (2-20) and (2-21) and then using Eq. (2-12), one gets the dispersion relation of an unbounded warm plasma:*

$$\begin{aligned} & W(1-Y^2 \cos^2 \theta) n^6 + [(-1+Y^2+X-XY^2 \cos^2 \theta)+2W(-1+Y^2 \cos^2 \theta+X)] n^4 \\ & + [(2-2Y^2+XY^2+XY^2 \cos^2 \theta-4X+2X^2) + W(1-Y^2 \cos^2 \theta-2X+X^2)] n^2 \\ & + (-1+Y^2+3X-XY^2-3X^2+X^3) = 0. \end{aligned} \quad (2-41.1)$$

where

$$\cos \theta = n_p/n, \quad (2-41.2)$$

θ being the angle between the propagation vector and \underline{B}_0 .

Replacing $\cos \theta$ in Eq. (2-41) by $(-\cos \theta)$ gives the same n^2 .

This shows that the refractive index surfaces are symmetric with respect to the plane $z = 0$.

*Another form of this equation with $(1/n)$ is given by Seshadri ⁽¹⁴⁾.

In order to put Eq. (2-41) into a more convenient form for the analysis of a waveguide one can use Eqs. (2-40.6) and (2-42) and get

$$\begin{aligned}
 & W n_t^6 + \{ [-1+X+Y^2] + W[-2+2X+(3-Y^2)n_p^2] \} n_t^4 \\
 & + \left\{ \begin{aligned} & [(2-4X+2X^2+XY^2-2Y^2) + (-2+2X-XY^2+2Y^2)n_p^2] \\ & + W[(1-2X+X^2)+(-4+4X+2Y^2)n_p^2 + (3-2Y^2)n_p^4] \end{aligned} \right\} n_t^2 \\
 & + \left\{ \begin{aligned} & [(-1+3X-3X^2+X^3-XY^2+Y^2) + (2-4X+2X^2+2XY^2-2Y^2)n_p^2] \\ & + (-1+X-XY^2+Y^2)n_p^4 \\ & + W[(1-2X+X^2-Y^2)n_p^2 + (-2+2X+2Y^2)n_p^4 + (1-Y^2)n_p^6] \end{aligned} \right\} = 0 \quad (2-42)
 \end{aligned}$$

which can also be written as

$$\begin{aligned}
 & W(1-Y^2)n_p^6 + \left\{ \begin{aligned} & [(-1+X-XY^2+Y^2)] \\ & + W[(-2+2X+2Y^2) + (3-2Y^2)n_t^2] \end{aligned} \right\} n_p^4 \\
 & + \left\{ \begin{aligned} & [(2-4X+2X^2+2XY^2-2Y^2) + (-2+2X-XY^2+2Y^2)n_t^2] \\ & + W[(1-2X+X^2-Y^2) + (-4+4X+2Y^2)n_t^2 + (3-Y^2)n_t^4] \end{aligned} \right\} n_p^2 \\
 & + \left\{ \begin{aligned} & [(-1+3X-3X^2+X^3-XY^2+Y^2)+(2-4X+2X^2+XY^2-2Y^2)n_t^2 + (1+X+Y^2)n_t^4] \\ & + W[(1-2X+X^2) + (1-2X+X^2)n_t^2 + (-2+2X)n_t^4 + n_t^6] \end{aligned} \right\} = 0 \quad (2-43)
 \end{aligned}$$

Equations (2-41), (2-42) and (2-43) are but three different expressions of the same dispersion relation. Eq. (2-41) is given in the polar coordinates (n, θ) whereas (2-42) and (2-43) are in the rectangular coordinates (n_t, n_p) . All these three equations are of the form

$$G_3 t^3 + G_2 t^2 + G_1 t + G_0 = 0 \quad (2-44)$$

where t can be n^2 , n_t^2 or n_p^2 and G_i 's are polynomials of the parameters $\cos^2 \theta$, n_p^2 or n_t^2 respectively.

The roots of Eq. (2-44) can be expressed explicitly with well known formulae.

Since in practice $W \ll 1$ which implies that

$$G_3 \ll 1 \quad (2-45)$$

some approximations for the roots of Eq. (2-44) can be made. Because all G 's contain a term of at most the first degree in W one writes G_i ($i = 0, 1, 2, 3$) as

$$G_i = G_{i0} + W G_{iW} \text{ with } G_{30} = 0$$

Considering inequality (2-45), in the regions of θ where

$$|W n_{1,2}^2| \ll 1 \quad (2-46.1)$$

and

$$|(n_1^2 - n_2^2)/n_1^2| \neq \text{arbitrarily small.} \quad (2-46.2)$$

The roots of dispersion relations which is summarized with Eq. (2-44) reduce to simpler expressions as given below:

$$\left. \begin{aligned} t_1 &= [-G_{10} + (G_{10}^2 - 4G_{20}G_{00})^{1/2}]/(2G_{20}) + O(G_3^2) \\ t_2 &= [-G_{10} - (G_{10}^2 - 4G_{20}G_{00})^{1/2}]/(2G_{20}) + O(G_3^2) \\ t_3 &= -(G_{20}/G_3) + O(G_3) \end{aligned} \right\} \text{ for } G_{10}^2 - 4G_{20}G_{00} > 0 \quad (2-46.3)$$

$$\left. \begin{aligned} t_1 &= [-G_{10} + j(4G_{20}G_{00} - G_{10}^2)^{1/2}]/(2G_{20}) + O(G_3^2) \\ t_2 &= [-G_{10} - j(4G_{20}G_{00} - G_{10}^2)^{1/2}]/(2G_{20}) + O(G_3^2) \\ t_3 &= -(G_{20}/G_3) + O(G_3) \end{aligned} \right\} \text{ for } G_{10}^2 - 4G_{20}G_{00} < 0 \quad (2-46.4)$$

If one neglects the terms $O(G_3^2)$, the expressions found for t_1 and t_2 become the same as those for a cold plasma. However, this happens only in regions of θ where inequalities (2-46.1) and (2-46.2) hold.

Thus, for θ not in these regions t_1 and t_2 are approximately equal to the solution of the equation

$$G_{20}t^2 + G_{10}t + G_{00} = 0 \quad (2-47.1)$$

which is found to be the dispersion relation of cold plasma, whereas t_3 is the solution of*

$$G_3t + G_{20} = 0 \quad (2-47.2)$$

For the cold plasma limit where $W = 0$, G_3 becomes zero and t_3 approaches infinity.

*These results agree with the solutions obtained by Seshadri⁽¹⁴⁾.

Using Eqs. (2-41) and (2-47) and assuming that G_{20} is not small, one finds for the third index surface

$$(n_3^2) = (1/W) [(1-Y^2)/(1-Y^2 \cos^2 \theta) - X].$$

This surface is asymptotic to the one

$$|\cos \theta| = 1/Y \text{ for } Y > 1. \quad (2-48.1)$$

For some value of θ , say θ_1 which satisfies

$$\cos^2 \theta_1 = (-1+Y^2+X)/(XY^2) \quad (2-48.2)$$

G_{20} becomes zero. Comparison of Eqs. (2-48.1) and (2-48.2) gives that for $Y > 1$

$$\theta_1 < \theta_0.$$

The cold plasma refractive index surfaces go to infinity for $\theta = \theta_1$ if

$$1) \quad X \geq 1; \quad Y^2 \geq 1 \quad (2-49.1)$$

or if

$$2) \quad X \leq 1; \quad Y^2 \leq 1; \quad Y^2 + X \geq 1. \quad (2-49.2)$$

For the compressible fluid model, however, very large values of n for $\theta = \theta_1$ can be obtained and are approximately given by

$$n^4 = X(1-X)(-1+2X+Y^2)/[W(1-Y^2)]; \quad |n| \gg 1. \quad (2-50)$$

This expression is positive for

$$I. \quad Y^2 \geq 1; \quad X > 1 \quad (2-51.1)$$

$$II. \quad Y^2 \geq 1; \quad Y^2 - 2X > 1; \quad X < 0; \quad (2-51.2)$$

$$\text{III} \quad -1 \leq Y^2 \leq 1; 0 < X < 1; Y^2 - 2X > 1; \quad (2-51.3)$$

$$\text{IV} \quad Y^2 \leq -1; X < 0; \quad (2-51.4)$$

$$\text{V} \quad Y^2 \leq -1; X > 1; Y^2 - 2X < 1 \quad (2-51.5)$$

These regions are shown in Fig. (1-1).

In the (X, Y^2) plane, the union of the regions given by inequalities (2-51) include the regions given by inequalities (2-49) except for some boundary lines. Excluding some small portions which we are going to discuss, in the latter regions n^4 of Eq. (2-50) is a large positive number.

The right hand side of Eq. (2-50) becomes zero on the line $X = 1$ and indeterminate at $X = 0, Y^2 = 1$. Therefore, in the regions described below, the roots of Eq. (2-50) cannot be very large:

1. The area between the lines

$$X = \pm \epsilon + 1 \quad (2-52.1)$$

where

$$|\epsilon/[W(1-Y^2)]|^4 \gg 1; \epsilon > 0;$$

2. The area which is inside the lower branch of the hyperbola

$$X(-1+2X+Y^2) - \delta W(1-Y^2) = 0$$

[which passes through the point $X = 0; Y^2 = 1$

where

$$\delta \gg 1$$

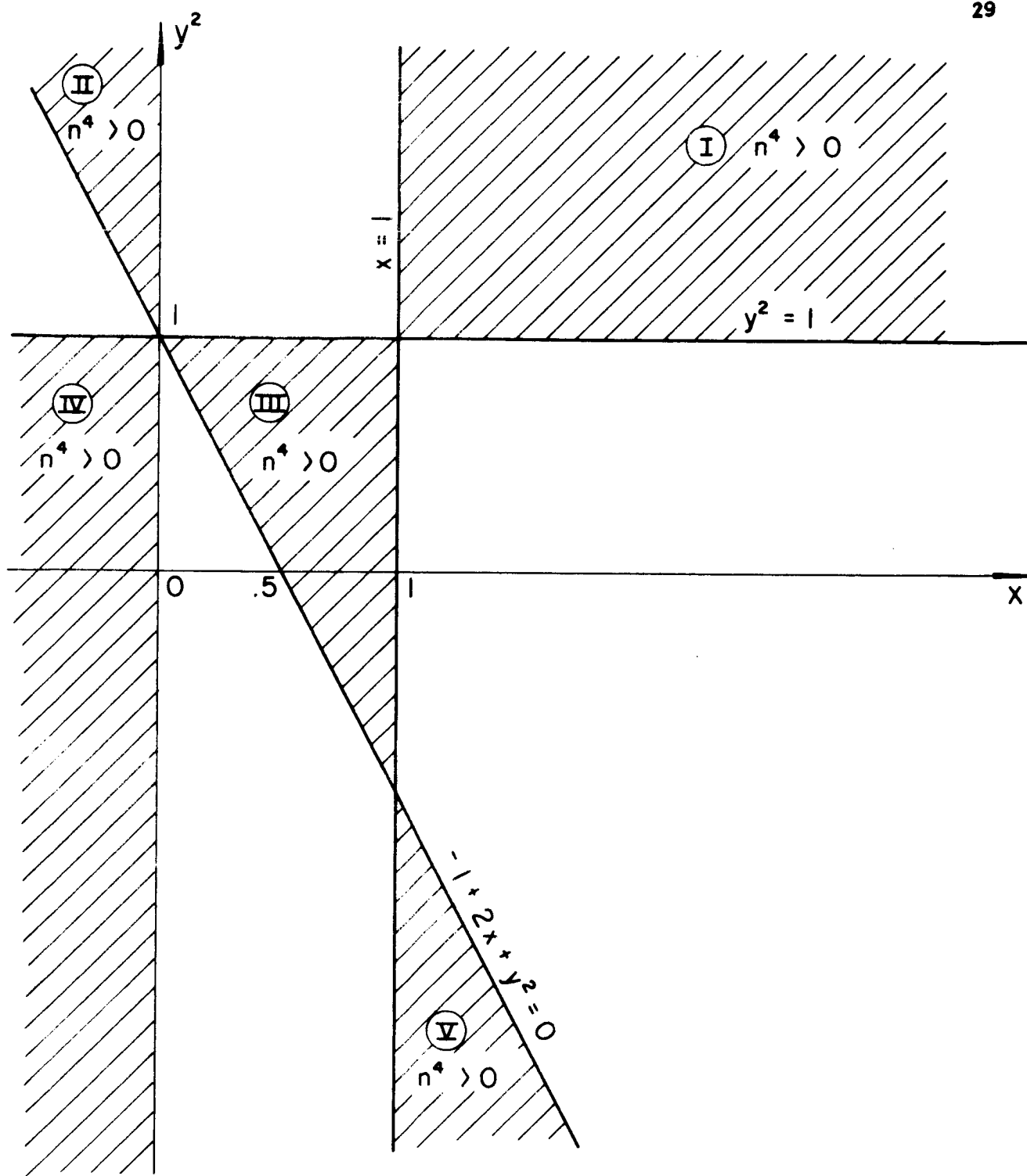


Figure (2-1) In various regions of $X-Y^2$ plane the sign of n^4 as given by Eq. (1-50).

or less restrictively the area left to the line

$$X = (1/\xi)$$

where

$$(W\xi)^4 \ll 1.$$

Eq. (2-50) indicates that in the regions of Eqs. (2-49) at angles where the cold plasma model gives infinitely large indices, the compressible fluid model gives two refractive indices of equal magnitude, one of which is real and the other one is purely imaginary.

$$\text{For } Y^2 \geq 1; \quad 1 \leq X < 1 - \epsilon$$

and $Y^2 \leq 1; \quad 1 - \epsilon < X < 1$, the angle θ_1 is very small. This angular region will be considered in Section 4 by using the Boltzmann theory and it will be shown that for small values of θ , n cannot be arbitrarily large and real.

At $\theta = \theta_1$, the third root of the dispersion relation approximates to

$$t \approx -G_{00}/G_{10}$$

The right hand side of this equation is equal to the solution for the smallest index surface for zero temperature.

For $Y^2 = 1 - X$, θ , becomes $\pi/2$ and the dispersion surfaces will approximate to that shown in Fig. (2-2).

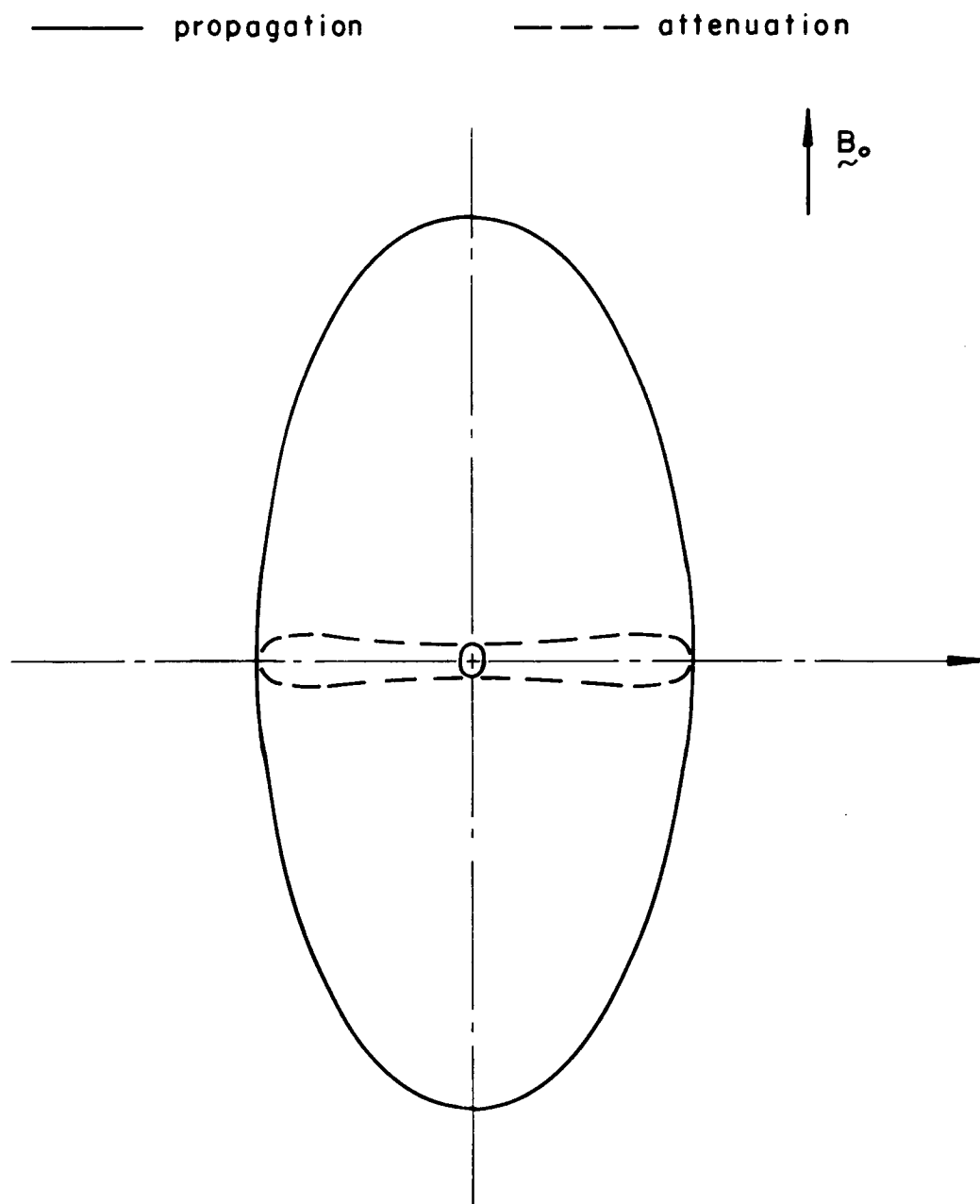


Figure (2-2) Dispersion surfaces for warm plasma with $Y^2 = 1 - X$

For $\gamma^2 = 1$, the targets refractive index become approximately equal to

$$n_3 = j \sqrt{X \left(\frac{1}{W} + \frac{2}{\sin^2 \theta} \right)}$$

The second refractive index is approximately equal to the larger refractive index of the cold plasma results except for $\theta \ll 1$. For $\theta \ll 1$, n_2 becomes

$$n_2 \approx \sqrt{\frac{1-X}{WX}}$$

On the other hand, n_1 only slightly deviates from the cold plasma results for all values of θ .

Figure (1-3) shows the refractive index surfaces for the compressible fluid model at $\gamma^2 = 1$ and three regions of X , namely

$X < 1$; $1 < X < 2$ and $2 < X$.

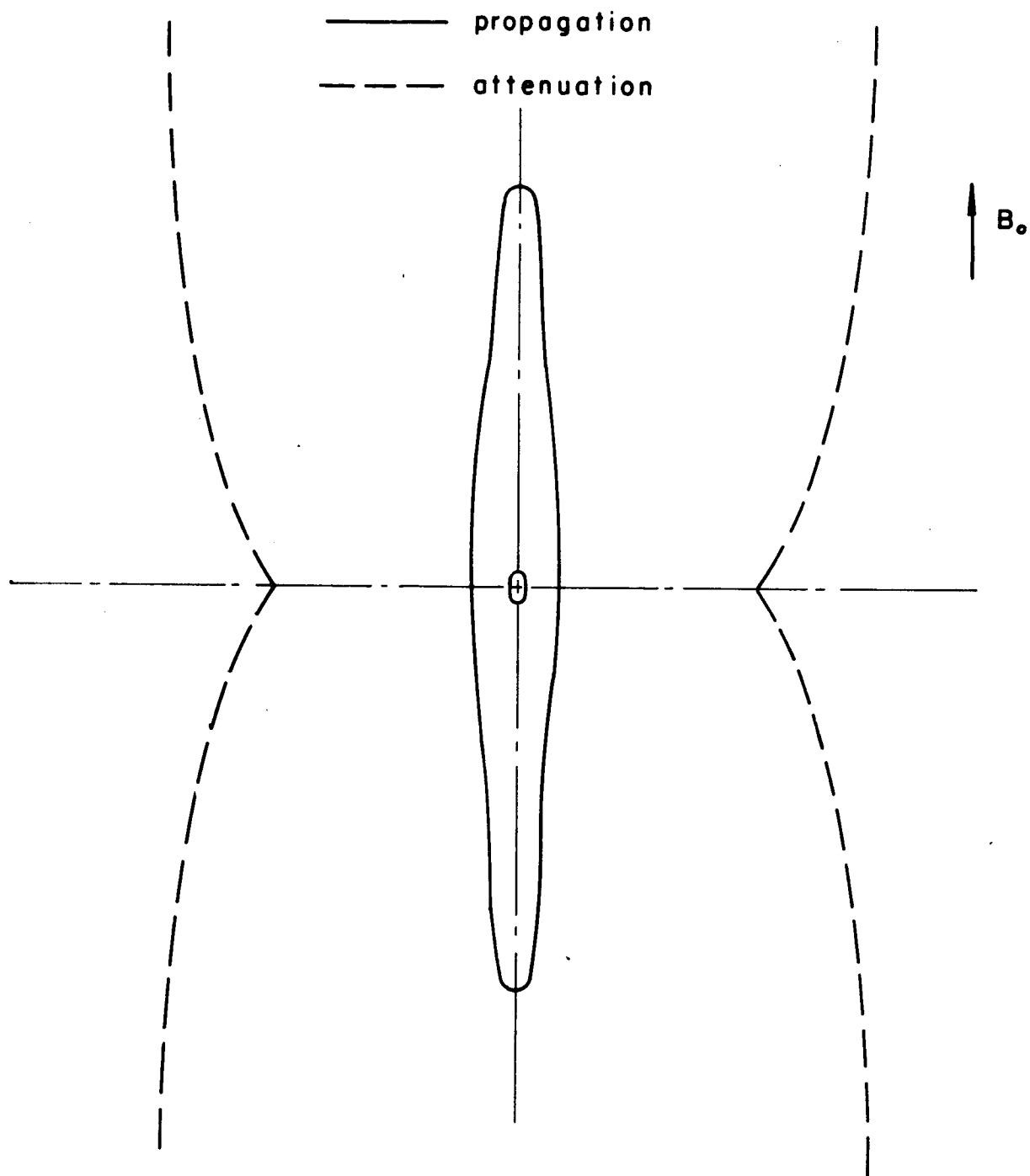


Figure (2-3) Dispersion surfaces for warm plasma with $Y = 1$

a) $X < 1$

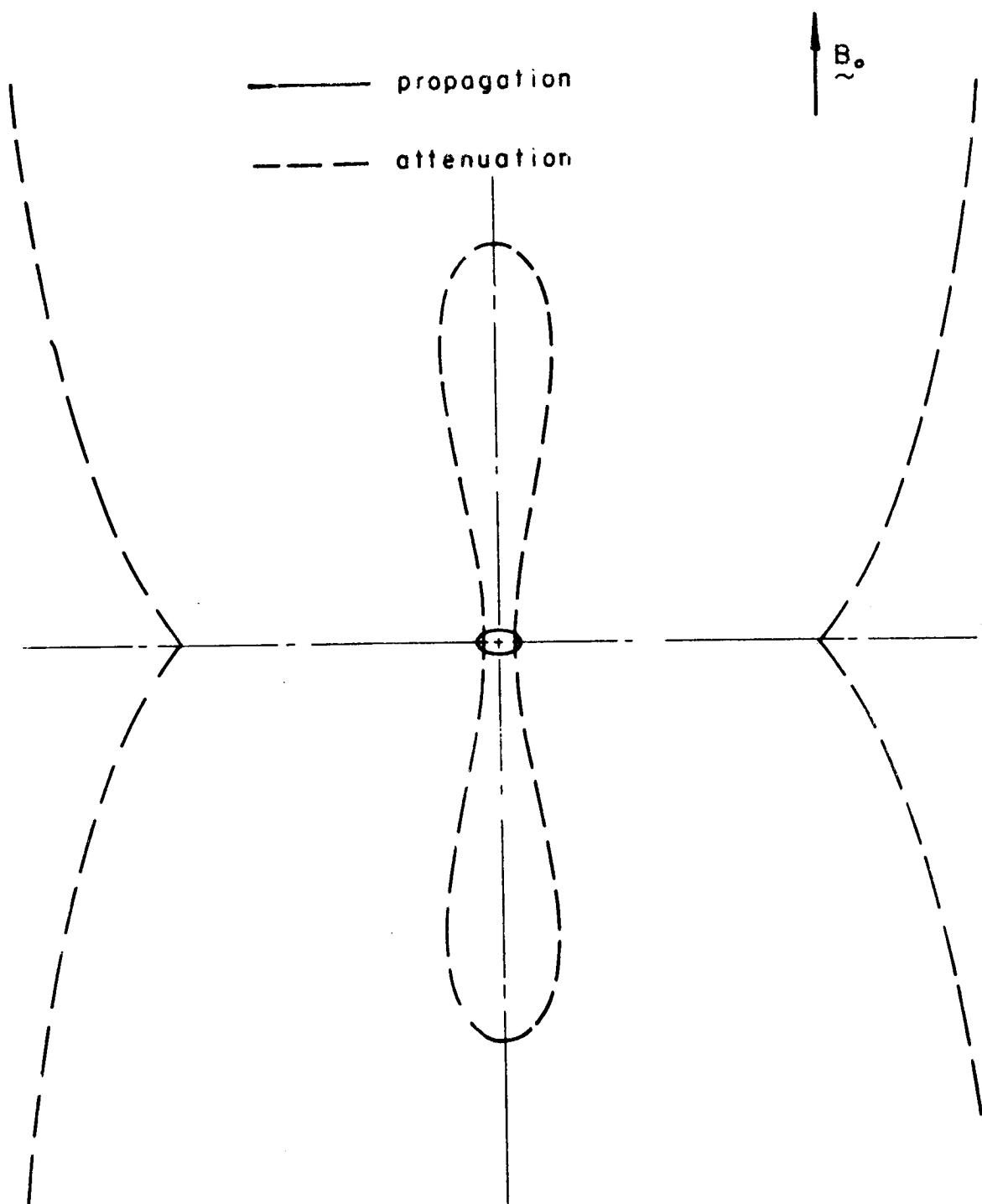


Figure (2-3) Continued

b) $1 < X < 2$

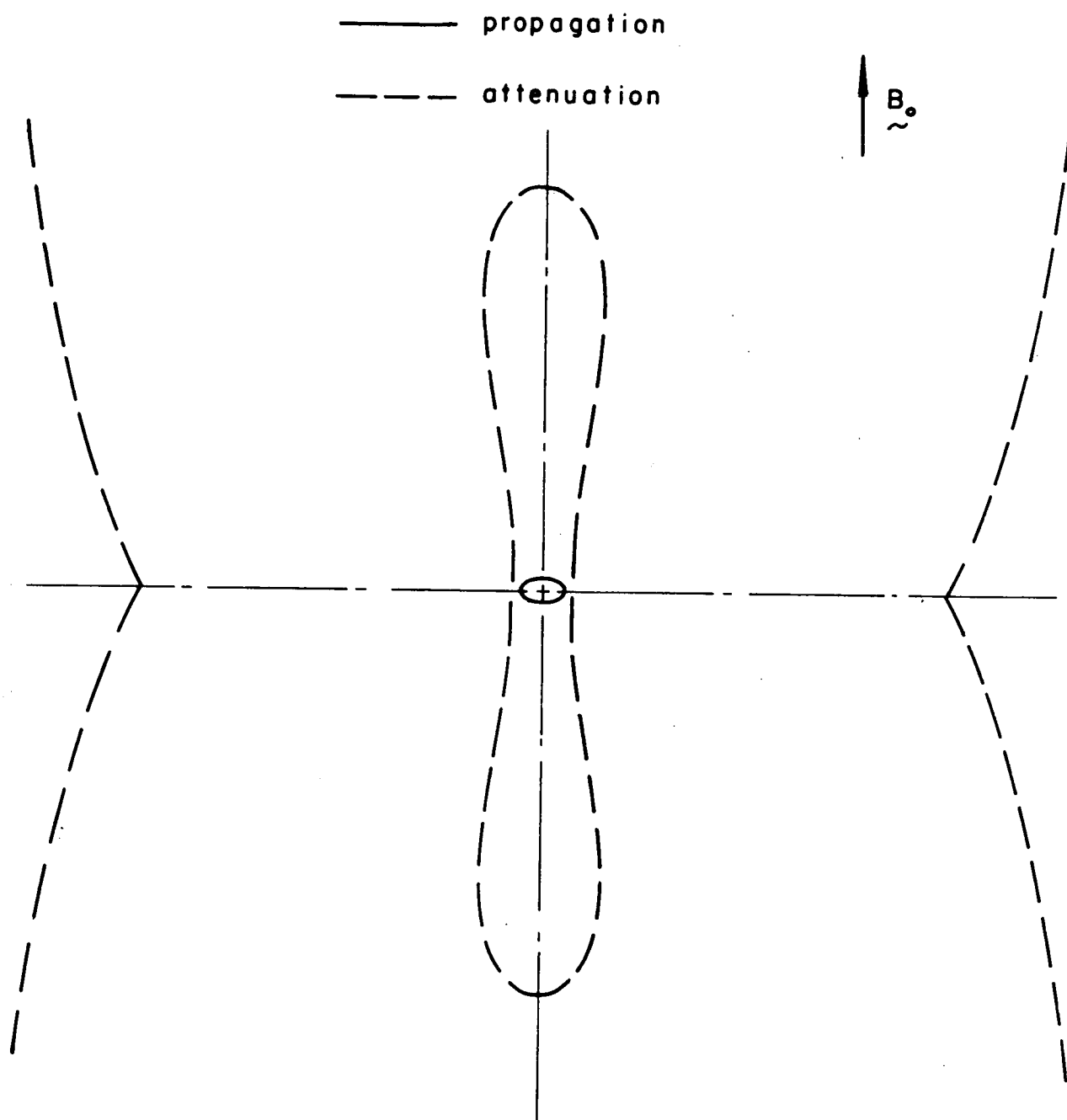


Figure (2-3) Continued

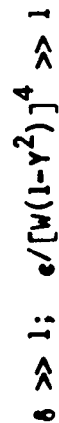
c) $2 < X$

For $Y^2 = (1-X)^2$, one of the propagation indices become zero, as it is for the cold plasma case.

Applying the above informations about the three roots of the dispersion equation a qualitative plot of refractive index surfaces for the compressible fluid plasma can be made as shown in Fig. (1-4). In this figure emphasis is given to the plot of the two largest index surfaces. The smallest one, on the other hand approximates to the inner surface of the cold plasma case for every value of X and Y^2 . Since the cold plasma results are well known, some of the forms which this surface has are not included in Fig. (2-4).

2.4 A Modification of the Warm Plasma Model

Since in hot plasma there may exist some electrons, the thermal velocities of which are very close to the wave velocity, a very strong interaction with the wave may be expected. This kind of effect cannot be found in an analysis based on the fluid model since in that model only averages over all velocities are considered. Therefore, for a more precise study, particularly for the case of resonance, the Boltzmann theory should be followed. Moreover, the validity of the fluid model is restricted to moderately high temperature; because in the truncation of the transport equations, which are taken as the basic equations for the study of the fluid model, it is implied that the thermal velocity $\sqrt{kT/m}$ is much smaller than the velocity of light in vacuum.



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In this section, the dielectric dyadic \underline{K} as studied in Section 3 will be reconsidered by using the Boltzmann's equation (2-34). Let us consider only the first order perturbation in all the quantities due to the applied a.c. field; then following the approach of Allis, Buchsbaum and Bers⁽³⁾, in the Fourier transformed domain we have for the matrix (K)

$$\underline{K} = \underline{1} - j \underline{\sigma}_r \quad (2-53)$$

$$\underline{\sigma}_r = \underline{U}^{-1} \underline{\sigma}_r^1 \underline{U} \quad (2-54.1)$$

$$\underline{U} = (1/\sqrt{2}) \begin{bmatrix} 1 & -j & 0 \\ 1 & j & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \quad (2-54.2)$$

$$(\sigma_r^1) = -\frac{X}{2Y} \int_{u=-\infty}^{\infty} du \int_{w=-\infty}^{\infty} w dw \int_{\phi=0}^{2\pi} d\phi \int_{\phi'=-\infty}^{\infty} \exp\{-j[a(\phi-\phi')-b(\sin\phi-\sin\phi')]\} [F_{ij}(\phi, \phi', w)] d\phi' \quad (2-54.3)$$

$$[F_{ij}(\phi, \phi', w)] = \begin{bmatrix} w d_w f_o \exp(j\phi - \phi') & w d_w f_o \exp(j\phi + \phi') & (2w)^{1/2} d_u f_o \exp(j\phi) \\ w d_w f_o \exp(-j\phi - j\phi') & w d_w f_o \exp(-j\phi + j\phi') & (2w)^{1/2} d_u f_o \exp(-j\phi) \\ (2u)^{1/2} d_w f_o \exp(-j\phi') & (2u)^{1/2} d_w f_o \exp(j\phi') & 2u d_u f_o \end{bmatrix} \quad (2-54.4)$$

where $f_o(w, u)$ is the velocity distribution function of the electron gas at equilibrium, without the applied a.c. field; u and w are the longitudinal and transverse components of electron velocities respectively,

$$a = (w - ju - uk_o n_p) / \omega_H;$$

$$\text{and } b = w k_o n_t / \omega_H.$$

By assuming f_o to be Maxwellian⁽³⁾, i.e.:

$$f_o = [m^3 / (8\pi^3 k^3 T^3)]^{1/2} \exp[-(w^2 + u^2)m / (2kT)] \quad (2-55)$$

one can find slightly simpler approximate expressions for the elements of the matrix (K) for certain values of θ provided that

$$|n|^2 k_o^2 kT / (m\omega_H^2) \gg 1 \quad (2-56.1)$$

and

$$\text{Re}(n^2 / |n|^2) > 0 \quad (2-56.2)$$

Assuming inequalities (2-56) are satisfied, then

$$1^0) \text{ for } |\theta| \ll 1$$

$$(K) = \begin{bmatrix} 1 - j(s/n) \cos^2 \theta & 0 & j(s/n) \cos \theta \sin \theta \\ 0 & 1 - j(s/n) & 0 \\ j(s/n) \cos \theta \sin \theta & 0 & 1 - j(s/n) \sin^2 \theta \end{bmatrix} \quad (2-57)$$

where

$$s = \chi(\pi e / W)^{1/2} = 0.966 \times 10^5 \chi T^{-1/2} \quad (2-58)$$

For the derivation of the above expression the reader is referred to Appendix 2.

2°) for $\theta = \pi/2$

$$(K) = \begin{bmatrix} 1 - (s/n) \cot(\pi/y) & 0 & 0 \\ 0 & 1 - (2s/n) \cot(\pi/y) & 0 \\ 0 & 0 & 1 - (s/n) \cot(\pi/y) \end{bmatrix} \quad (2-59)$$

The derivation of the above expression is also given in Appendix 2.

In deriving Eqs. (2-57) and (2-59) we have assumed that the collision frequency ν is zero. Therefore, in both cases (K) must be Hermitian.* This implies that in (K) of Eq. (2-57) n must be purely imaginary and in (K) of Eq. (2-59) n must be real.

For the first case, the above conclusion, however, contradicts the assumption made in (2-56.2) which requires that $\text{Re } n > |\text{Im } n|$. Because of this contradiction one concludes that inequalities (2-56) cannot be satisfied together. In other words n cannot be arbitrarily large if $|\arg n| < \pi/4$ (of course, including real n).

It may be recalled that the dispersion surface of (2-41) which is derived from the fluid equations has asymptotes at $|\cos \theta| = 1/\gamma$. Thus, for small θ , $\theta^2/2 \approx -1 + \gamma$; i.e. near gyroresonance n becomes an infinitely large real number at $\theta \ll 1$, in contrast to the results obtained above. Therefore, for a more correct result near gyroresonance the asymptotes should be replaced by the numerical solutions of Eqs. (2-53) and (2-54).

*See Appendix 1.

For the second case, let

$$s' = 2s \cot(\pi/Y); \quad (2-60)$$

then one obtains the dispersion relation at $\theta = \pi/2$ as follows:

$$\begin{aligned} n^7 - 0.5 s' n^6 - 2n^5 + 2.5n^4 - (0.75s'^2 + 1) n^3 \\ + 2s' n^2 - 1.25s'^2 n + 0.25 s'^3 = 0. \end{aligned} \quad (2-61)$$

The largest three real roots of this equation are computed and listed in Table (1-1) for various values of s' . In order to examine whether these roots are consistent with inequality (2-56.1), we first let s be expressed in terms of temperature:

$$s = 0.966 \times 10^5 X T^{-1/2} \quad (2-62)$$

Then inequality (2-56.1) can be written as

$$\frac{|n|}{s} \frac{X}{Y} \gg 1; \quad (2-63)$$

or equivalantly

$$|n T^{1/2} \times 10^{-5} Y^{-1}| \gg 1. \quad (2-64)$$

Therefore, in general, for given X , Y and T one can compute s and n from Eqs. (2-62), (2-60) and (2-61), as given in Table (1-1).

For those solutions of n which satisfy (2-63) or (2-64), they are consistent with the assumption in deriving Eq. (2-61) and must be correct solutions. Otherwise they should be discarded. In particular for some values of X , Y and T , three cases will be considered in the following.

TABLE (1-1)

s'	n_1	n_2	n_3	$ n_{\max} s'$
1000.	500.0099800	-010.03544600	-7.77026580	00000.50000998
100.	050.09860300	-004.73159400	-3.33578280	00000.50098603
10.	005.6948011	-002.42433580	-1.078128	00000.56948011
1.	-001.8381067	001.11019700	-0.15503451	00001.83810670
0.1	-001.883596	000.13886565	-0.01572040	00018.83598600
0.01	-001.8950423	000.66307124	-0.00157290	00189.50423000
0.001	-001.8962663	000.68789738	-0.00015730	01896.26630000
0.0001	-001.8963895	$-10^5 \times 1.5729802$	-----	18983.89500000
-0.0001	000.0000000	-----	-----	00000.00000000
-0.001	000.0000000	-----	-----	00000.00000000
-0.01	-001.8977820	000.71505412	0.00157306	00189.77820000
-0.1	000.0000000	-----	-----	00000.00000000
-1.	-002.1182282	001.30834450	0.15650215	00002.11822820
-10.	-005.8128064	002.33128050	1.09581670	00000.58128064
-100.	050.1005860	004.72230080	3.34049340	00000.50100586
-1000.	500.0100000	100.34847000	7.77062570	00000.50001000

1) If the frequency is near gyroresonance or its multiples, i.e.

$$\frac{1}{p} - \epsilon < Y < \frac{1}{p} + \epsilon, \quad (2-65)$$

$$p = \pm 1, \pm 2, \dots,$$

$$0 \leq \epsilon \ll 1,$$

then

$$|\cot \pi/Y| \gg 1.$$

Therefore from Eq. (2-60), in general, s^1 becomes very large, and the solution of n with largest absolute value can be approximated by

$$n = \frac{s^1}{2} + \frac{10}{s^1} - \frac{2.5}{s^{1/2}} + \frac{40}{s^{1/3}}; \quad (2-66)$$

i.e. $n \approx s^1/2$ as seen in the first and the last three rows in Table (1-1).

Of course, the validity of this root depends on T and ϵ which should satisfy the inequality (2-64). In case of exact gyroresonance, i.e. $\epsilon = 0$, $n = \infty$ for any non vanishing T .

Physically this may be explained in the following manner. In the absence of collisions, electrons, excited by an electromagnetic wave at their gyrofrequency (or its multiples), which is propagating in the plane of their orbits, become resonant. Under the stationary state, the wavelengths of the electromagnetic wave must approach zero, for, otherwise the electron orbits will grow infinitely large and will not be stationary.

However, once collisions are introduced, in the equations, one has to replace Y with $Y/(1-jZ)$. This in turn will cause s' to be replaced by

$$s' = [2.s/(1-jZ)] \cot [\pi(1-jZ)/Y].$$

Therefore, for

$$Y = 1/p, \quad p = \text{a nonzero integer}$$

one would solve for s'

$$s' = [(2js)/(1-jZ)] \coth (pZ\pi)$$

which is no longer infinitely large; hence, neither is $|n|_{\max}$

2) For the special case when

$$X \rightarrow 0$$

and Y^2 approaches 1 through the line $Y^2 = 1-X$ then $\lim_{\substack{X \rightarrow 0 \\ Y^2 = 1-X}} s' = (1.23) 10^5 T^{-1/2}$.

Thus in this particular case inequality (2-64) becomes

$$|n/s'| \gg 1. \quad (2-67)$$

This condition is satisfied as long as $|s'|$ is very small, that is for very high temperatures. Therefore, in this case the approximate solutions obtained by solving Eq. (2-61), namely those given in Table (1-1), are correct.

It is interesting to note that when $Y^2 = 1-X$, n becomes infinite at $\theta = \pi/2$ for the cold plasma model in contrast to the above results.

3) If the frequency is not in the neighborhood of gyroresonance (or its multiples) and half of its odd multiples, the ratio $|s/s'|$ will be not very large. Hence, inequality (2-63) will be satisfied if

$$0 < s' \ll 1 \quad (2-68)$$

which corresponds to very high temperature or if

$$X/Y \gg 1. \quad (2-69)$$

In either case the approximate solution given by (2-61) is correct although n is not very large, as shown in Table (1-1) for $s' = 0.1$ to -0.1 .

As a conclusion to this section one can state that for the ranges of θ under consideration n can have arbitrarily large real values only for $\theta = \pi/2$ provided that T is very large and the frequency is near gyroresonance (or its multiples).

III. FIELD EXPRESSIONS IN ANISOTROPIC PLASMA AND CHARACTERISTIC EQUATION OF THE CIRCULAR WAVEGUIDE FILLED WITH LONGITUDINALLY MAGNETIZED PLASMA

In the last chapter we have discussed the dispersion relation in a magnetoplasma of various models. In essence this relation is a description of the wave number as a function of the angle of wave propagation, or the longitudinal wave number as a function of the transverse wave number or vice versa. For a waveguide not all wave numbers are permissible since the boundary conditions must be satisfied. Therefore, in the following we shall first determine the form of the field solution appropriate to the guide geometry; second obtain a characteristic equation which relates the dispersion relation to the boundary conditions and third investigate the electron temperature effect on the wave number and fields. Later in Chapter V this equation is numerically solved for two different models of the plasma.

3.1 Derivation of Field Expressions in a Cold Plasma

For the case of cold plasma, once the dispersion relation (2-24.3) is solved, the field can be determined using Eq. (2-18). Inserting Eq. (2-32) into Eqs. (2-31.7) and (2-31.8) one finds

$$C_{1j} = -j k_{op} (-v_j^2 + k_{xx} - k_{op}^2) d_x + j k_{op} k_{xy} d_y \quad (3-1)$$

$$C_{2j} = -j k_{op} (-v_j^2 + k_{xx} - k_{op}^2) d_y - j k_{op} k_{xy} d_x \quad (3-2)$$

Defining

$$S_j = (-v_j^2 + k_{xx} - k_{op}^2) \quad (3-3)$$

$$L_j = (-k_{op}^2 + k_{xx})^2 + k_{xy}^2 - v_j^2(-k_{op}^2 + k_{xx}) \quad (3-4)$$

and making use of Eqs. (2-17.8) and (2-17.9) one has

$$\tilde{E}_j = -j k_{op} M_j \cdot \nabla \pi_j ; \quad j = 1, 2 \quad (3-5.1)$$

where π_j is a solution to the equation

$$(\nabla_t^2 + v_j^2) \pi_j = 0$$

and the matrix (M_j) is given by

$$(M_j) = \begin{bmatrix} S_j & -k_{xy} & 0 \\ k_{xy} & S_j & 0 \\ 0 & 0 & -L_j / (k_{op}^2) \end{bmatrix} \quad (3-5.2)$$

For the case where B_0 is parallel to the axis of the circular waveguide the solution π_j for finite fields inside the guide has the following form

$$\pi_j = [\exp(jm_1 \varphi)] [\exp(vz)] J_{m_1}(v_j r) \quad (3-5.3)$$

where $J_{m_1}(v_j r)$ is a Bessel function of the first kind.

Taking the Fourier transform of Eq. (2-6) with respect to the variables t and z and making use of Eq. (3-5) one obtains the expression^{*} for \tilde{H}_j :

$$\tilde{H}_j = j(k_o/w\mu_o) \tilde{Q}_j \cdot \nabla \pi_j \quad (3-5.4)$$

where

$$(\tilde{Q}_j) = (k_o)^{-1} \begin{bmatrix} k_o^2 n_p^2 k_{xy} & L_j + k_o^2 n_p^2 S_j & 0 \\ -k_o^2 n_p^2 S_j - L_j & k_o^2 n_p^2 k_{xy} & 0 \\ 0 & 0 & -k_{xy}^2 v_j^2 \end{bmatrix} \quad (3-5.6)$$

For $n_p = 0$, i.e. at cut-off one has either

$$1) \quad v_1^2 = k_{zz}$$

or

$$2) \quad v_2^2 = k_{xx} + (k_{xy}^2 / k_{xx}).$$

For case (1) all the cofactors ($A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$) except C_3 become zero. Therefore, from Eqs. (2-17.8) and (2-17.9) one concludes that $E_{x1} = E_{x2} = 0$. For case (2), on the other hand, A_3, B_3, C_1, C_2 and C_3 are zero. Eqs. (2-17.2), (2-17.3), (2-17.5) and

*The dual of this result for a gyromagnetic medium coincides the one given by Epstein⁽⁸⁾.

(2-17.6) imply that $E_{z2} = 0$ as $n_p = 0$. These results are included in limit cases of Eqs. (3-5). For arbitrarily small n_p , one has, with the aid of Eqs. (2-24), (3-3), (3-4) and (3-5):

$$S_1 = 0(1) \quad (3-8.1)$$

$$L_1 = 0(1) \quad (3-8.2)$$

$$S_2 = 0(1) \quad (3-8.3)$$

$$L_2 = 0(n_p^2) \quad (3-8.4)$$

hence

$$E_{\sim t1} = 0(n_p), \quad (3-8.5)$$

$$E_{z1} = 0(1), \quad (3-8.6)$$

$$E_{t2} = 0(n_p), \quad (3-8.7)$$

$$E_{z2} = 0(n_p^2). \quad (3-8.8)$$

Also it can be shown that for $n_p \ll 1$,

$$H_{\sim t1} = 0(1), \quad (3-8.9)$$

$$H_{z1} = 0(n_p), \quad (3-8.10)$$

$$H_{\sim t2} = 0(n_p^2), \quad (3-8.11)$$

$$H_{z2} = 0(n_p). \quad (3-8.12)$$

This implies that for $n_p = 0$ case (1) corresponds to a TM mode with only E_z

and H_t non-vanishing; whereas case (2) corresponds to a TE mode with only E_t/n_p and H_z/n_p non-vanishing.

For the general case where $n_p \neq 0$, Eq. (3-5) shows that H_{zj} can be expressed in terms of E_{zj} :

$$H_{zj} = -j[L_j/(k_o n_p)] [1/(k_{xy} v_j^2)] E_{zj}. \quad (3-7)$$

In contrast when $B_o = 0$, the waves associated with $j = 1, 2$ reduce to TE and TM modes and E_z and H_z become uncoupled.

For the case of small B_o , i.e. $Y \ll 1$, the following approximations can be made:

$$v_{1,2}^2 = k_o^2 (n_{ot}^2 \mp \beta)$$

$$s_{1,2} = \pm k_o^2 \beta$$

$$L_{1,2} = \pm k_o^4 n_{to}^2 \beta$$

$$(M_{1,2}) = j k_o^2 \alpha \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mp k_o^2 \beta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -n_{to}^2/n_p^2 \end{bmatrix}$$

$$(Q_{1,2}) = j k_o^3 \alpha n_p^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n_{to}^2/n_p^2 \end{bmatrix} \mp k_o^3 \beta n_o^2 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{where } n_{to}^2 = 1 - X - n_p^2$$

$$n_o^2 = 1 - X$$

$$\alpha = XY$$

$$\beta = [X(n_p^4 - 1 + X + 2Xn_p^2) / (1 - X)]^{1/2}.$$

Thus, from this result and Eqs. (3-5.1) and (3-5.4) it can be seen that once B_{ω} is introduced into an ionized medium, each wave becomes a combination of TE and TM modes which are coupled through Y as given by the first and second terms of the matrices $(M_{1,2})$ and $(Q_{1,2})$ in the above equations.

The characteristic equation of the waveguide will now be formed by imposing the boundary conditions. In the case of a cold plasma the boundary conditions require that the tangential component of the electric field vanish on the conducting boundary; i.e.

$$E_z(P) = 0 \quad (3-9.1)$$

and

$$\underline{E}_t(P) \times \underline{\hat{n}} = 0 \quad (3-9.2)$$

where P is a point on the guide surface; \underline{E}_t is the transverse component of \underline{E} and $\underline{\hat{n}}$ is a unit normal vector at the conducting wall.

Since Eqs. (3-5) imply that in the waveguide the electric field generally has a tangential component in the transverse plane as

well as an axial component and those components are not proportional, in order to satisfy Eqs. (3-9) generally one has to have at least two waves of one mode propagating in the waveguide, presumably, one corresponding to the subindex $j = 1$, namely the ordinary wave and the other one corresponding to the subindex $j = 2$, namely the extraordinary wave. Both of these waves have to have the same propagation constant j_y along the z axis, because the boundary conditions have to be satisfied for all values of z .*

In this case the total field in the wave guide will be composed of

$$\underline{E} = \delta_1 \underline{E}_1 + \delta_2 \underline{E}_2$$

and

$$\underline{H} = \delta_1 \underline{H}_1 + \delta_2 \underline{H}_2$$

and the ratio between δ_1 and δ_2 will be determined by the boundary conditions. This is discussed in the next section.

3.2 Derivation of the Characteristic Equation for the Waveguide Filled with Cold Plasma

Let the radius of the circular guide be r_0 . Then the boundary

*However, for propagation transverse to the dc magnetization, i.e. $\gamma=0$, waves degenerate into two. One of them has electric field parallel to $B_0 \hat{z}$ and the other one has electric field transverse to B_0 (See Eqs. (3-8)). This enables us to find the zeroth order modal solutions of cold plasma filled rectangular guides with B_0 perpendicular to one of the guide walls.

For an axially magnetized circular waveguide, when $n_p = 0$, the ordinary and extraordinary waves become uncoupled. One of them is a TE wave and the other one is a TM wave each of which satisfy boundary conditions separately.

conditions given by Eq. (3-8) reduces to $E_z = E_\varphi = 0$ at $r = r_o$. From these conditions, one obtains

$$[S_1(\partial/\partial r_o)\pi_1(r_o) - j k_{xy}(m/r_o)\pi_1(r_o)]\delta_1 + [S_2(\partial/\partial r_o)\pi_2(r_o) - j k_{xy}(m/r_o)\pi_2(r_o)]\delta_2 = 0 \quad (3-10.1)$$

$$[jL_1 \pi_1(r_o)/(k_o n_p)]\delta_1 + [jL_2 \pi_2(r_o)/(k_o n_p)]\delta_2 = 0 \quad (3-10.2)$$

For the nontrivial solution of the field (i.e. $\delta_1 \neq 0$, $\delta_2 \neq 0$) the determinant of the coefficients in (3-10.1) and (3-10.2) must be equal to zero, namely:

$$\begin{vmatrix} S_1\pi_1'(r_o) - j k_{xy}(m/r_o)\pi_1(r_o) & S_2\pi_2'(r_o) - j k_{xy}(m/r_o)\pi_2(r_o) \\ j L_1 \pi_1(r_o)/(k_o n_p) & j L_2 \pi_2(r_o)/(k_o n_p) \end{vmatrix} = 0. \quad (3-11)$$

This characteristic equation contains the unknowns n_p and $v_{1,2}$ and can be solved in conjunction with the dispersion relation (2-24.3). The roots n_p give propagation constants of the modal fields that can exist in the waveguide. Once n_p is determined, the functions π_j become known and from Eq. (3-10) one can find the ratio of (δ_1/δ_2) for any root n_p and this will complete the determination of the modal waves.

For $n_p = 0$, i.e. at cut-off we have seen that \tilde{E}_t and E_z do not belong to the same wave, i.e., one belongs to the ordinary wave and

the other one to the extraordinary wave. Therefore, now the boundary conditions can be fulfilled by each wave separately. For the TM wave which has

$$v_1^2 = k_{zz}$$

the following condition

$$J_m[(k_{zz})^{1/2} k_o r_o] = 0$$

has to be satisfied; and for the TE wave which has

$$v_2^2 = k_{xx} + (k_{xy}^2 / k_{xx})$$

the characteristic equation becomes

$$[m/(k_o r_o)](K_{xx} - j K_{xy}) J_m(n_{t2} k_o r_o) - K_{xx} n_{t2} J_{m+1}(n_{t2} k_o r_o) = 0$$

$$n_{t2} = [(X^2 - 2X + 1 - Y^2)/(1 - Y^2 - X)]^{1/2}.$$

The above two equations will determine the cut-off frequencies for TM and TE waves.

3.3 Derivation of Field Expressions in a Warm Plasma

In Chapter II we obtained the dispersion relation for a plane wave in a warm anisotropic plasma. Since the dielectric dyadic in such a medium is a function of both the frequency (ω) and the wave vector \underline{k} , and since the formulas determined are valid only for \underline{k} in the xz plane,

the characteristic equation cannot be so simply determined as for the cold plasma case in the last section. To proceed we may consider the field in the guide as a superposition of infinitely many plane waves.

Let us consider a wave propagating in the medium with a certain refractive index component n_p along the z axis. Using Eqs. (2-42), (2-44) and (2-45) one can find the three values of n_t which correspond to that particular value of n_p . Let n_{t1} , n_{t2} , and n_{t3} be those three values. The transform theory tells us that, together with the dispersion relation (2-42), corresponding to the given n_p one can have infinitely many propagating plane waves, the transverse component of their propagation constants having the magnitude of one of the three values of n_t . Let us focus our attention on only those waves which for example have their transverse propagation vector with magnitude $k_0 n_{t1}$.

In the transverse plane one can consider an interval of angle $(\phi, \phi + d\phi)$ and assume that one of the characteristic plane waves with the transverse propagation vector lying in this interval have a field strength $A(\phi)$, where $A(\phi)$ is some complex function of the real variable ϕ . Since $A(\phi)$ must be a periodic function of ϕ with the period 2π one can expand $A(\phi)$ in a Fourier series as

$$A(\phi) = \sum_{m=-\infty}^{\infty} A_{m0} \exp(jm\phi) \quad (3-12)$$

Let us now consider a certain term of this series, for example $m = m_p$ and then determine the field at a point P with cylindrical coordinates (r, ϕ, z) . Figure (3-1) shows the geometry of the cross section of a guide.

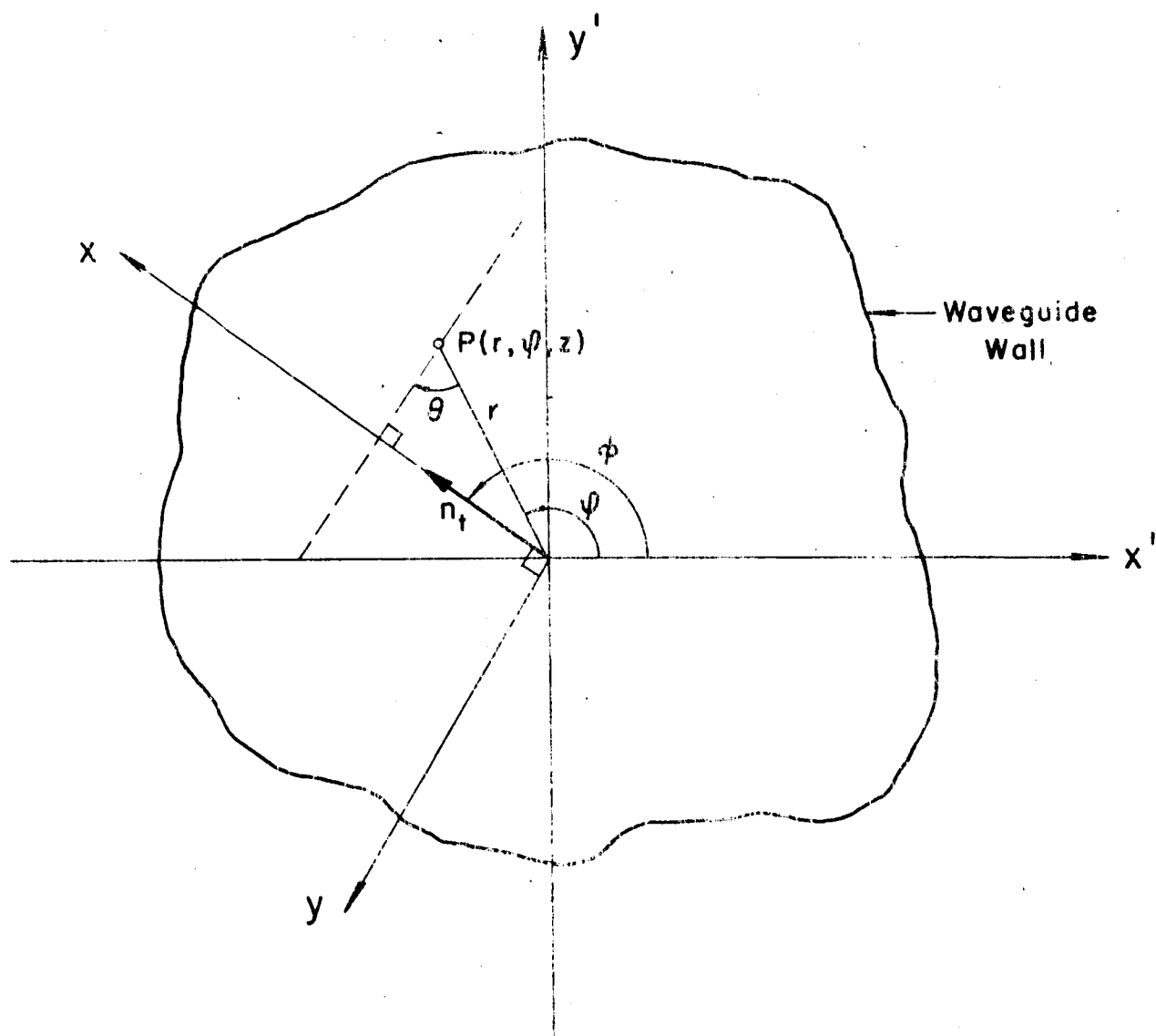


Figure (3-1) The cross section of an arbitrary waveguide

Let us consider a transverse propagation vector which makes an angle Φ with the x^1 axis of the transverse plane. According to Equation (3-5) the contribution of this particular wave to the E_z at P will be

$$dE_z = A_{m_1 o} \exp(jm_1 \Phi) \exp(j k_{o n_t} r \sin \theta) \exp(-jk_{o n_p} z) d\Phi \quad (3-13)$$

where θ is the angle between the position vector of P and the wave front.

$$\text{Since } \Phi = \varphi + 3\pi/2 - \theta$$

if one defines C_o with the equation

$$A_{m_1 o} \exp(jm_1 3\pi/2) = k_{o n_t} C_3(n_p) [C_o / (2\pi)] d\Phi \quad (3-14)$$

where for a fixed φ

$$d\Phi = -d\theta \quad (3-15)$$

and C_3 is defined with Eq. (2-18.3).

Integrating over all values of Φ one obtains for E_{zj}

$$E_{zj} = -k_{o n_t j} C_{3j} C_{oj} \exp(-jk_{o n_p} z) \exp(jm_1 \varphi) \times \quad (3-16)$$

$$(1/2\pi) \int_{\theta=0}^{2\pi} \exp(jk_{o n_t j} r \sin \theta - m_1 \theta) d\theta$$

$$\text{or } E_{zj} = -C_{oj} C_{3j} k_{o n_t j} \exp(-jk_{o n_p} z + jm_1 \varphi) \times J_{m_1}(k_{o n_t j} r). \quad (3-17)$$

where, as before, the subscript j is added to denote the j-th characteristic wave of the medium.

In obtaining the solution given above, the Bessel function is chosen as solution in order to have the field finite in the guide.

The r - and φ - components of the electric field at point P can be found from the following equations:

$$dE_{rj} = -\sin\theta dE_{xj} + \cos\theta dE_{yj} \quad (3-18.1)$$

and

$$dE_{\varphi j} = -\cos\theta dE_{xj} - \sin\theta dE_{yj} \quad (3-18.2)$$

where

$$dE_{xj} = -C_{oj} k_{on_{tj}} C_{1j} \exp(-jk_{on_p} z + m_1 \varphi) \\ \times (1/2\pi) \exp[j(k_{on_{tj}} r \sin\theta - m_1 \theta)] d\theta \quad (3-18.3)$$

$$dE_{yj} = -C_{oj} k_{on_{tj}} C_{2j} \exp(-jk_{on_p} z + m_1 \varphi) \\ \times (1/2\pi) \exp[j(k_{on_{tj}} r \sin\theta - m_1 \theta)] d\theta \quad (3-18.4)$$

and C_{1j} and C_{2j} are defined with Eqs. (2-18.1) and (2-18.2) respectively.

Integrating over θ again one finds

$$E_{rj} = C_{oj} k_{on_{tj}} [\exp(-jk_{on_p} z + jm_1 \varphi)] \\ \times (1/2) \left[(-jC_{1j} - C_{2j}) J_{m_1-1}(k_{on_{tj}} r) \right. \\ \left. + (jC_{1j} - C_{2j}) J_{m_1+1}(k_{on_{tj}} r) \right] \quad (3-19.1)$$

and

$$E_{\varphi j} = C_{oj} k_{on} n_{tj} [\exp(-j k_{on} n_p z + j m_1 \varphi)]$$

$$(1/2) \left[\begin{aligned} & (C_{1j} - j C_{2j}) J_{m_1-1}(k_{on} n_{tj} r) \\ & + (C_{1j} + j C_{2j}) J_{m_1+1}(k_{on} n_{tj} r) \end{aligned} \right] \quad (3-19.2)$$

or

$$E_{rj} = j C_{oj} [\exp(-j k_{on} n_p z + j m_1 \varphi)]$$

$$[C_{2j} (m_1/r) J_{m_1}(k_{on} n_{tj} r) - C_{1j} d_r J_{m_1}(k_{on} n_{tj} r)] \quad (3-20)$$

and

$$E_{\varphi j} = -j C_{oj} \exp(-j k_{on} n_p z + j m_1 \varphi)$$

$$[C_{1j} (j m_1/r) J_{m_1}(k_{on} n_{tj} r) + C_{2j} d_r J_{m_1}(k_{on} n_{tj} r)] \quad (3-21)$$

It may again be noted that C_{1j} , C_{2j} , C_{3j} in the above equations are the cofactors belonging to the j -th characteristic plane wave and are expressed by Eqs. (2-18.1), (2-18.2) and (2-18.3). They are

$$C_{1j} = k_{on}^4 n_p n_{tj} \{ F_j - 1 + n_j^2 + W [F_j (1 - F_j) - 2 F_j n_j^2 + 2 F_j^2] + W^2 F_j^2 (n_j^2 - Y n_p^2) \} \quad (3-22.1)$$

$$C_{2j} = j k_{on}^4 Y F_j n_p n_{tj} (1 - W) \quad (3-22.2)$$

$$C_{3j} = k_{on}^4 [1 - n_j^2 - n_p^2 + n_j^2 n_p^2 + F_j (-2 + n_j^2 + n_p^2 + X) + W F_j (n_j^2 + n_p^2 - 2 n_j^2 n_p^2 - X n_p^2)] \quad (3-22.3)$$

where

$$F_j = X/[1-Y^2+W(-n_j^2+Y^2n_p^2)] \quad (3-22.4)$$

and $(j = 1, 2, 3)$. It should be reminded that so far only the Fourier component for $m = m_1$ is considered.

In general one can define a potential function π_j as

$$\pi_j = \sum_{m=-\infty}^{\infty} C_{oj}(m) J_m(k_o n_{tj} r) \exp(-j k_o n_p z + j m \phi) \quad (3-23.1)$$

This implies that

$$\pi = \sum_{j=1}^3 \delta_j \pi_j \quad (3-23.2)$$

is a solution to the differential equation

$$(\nabla_t^2 + k_o^2 n_{t1}^2)(\nabla_t^2 + k_o^2 n_{t2}^2)(\nabla_t^2 + k_o^2 n_{t3}^2)\pi = 0 \quad (3-23.3)$$

Then one finds that

$$\vec{E}_j = -j k_o n_p M_j \cdot \nabla \pi_j \quad (3-23.4)$$

where

$$(M_j) = [1/(k_o n_p)] \begin{bmatrix} c_{1j} & -c_{2j} & 0 \\ c_{2j} & c_{1j} & 0 \\ 0 & 0 & c_{3j} n_{tj}/n_p \end{bmatrix} \quad (3-23.5)$$

Using the Fourier transform of Eq. (2-6) (with zero magnetic source current), one can find the expression of \tilde{H}_j in terms of π_j as

$$\tilde{H}_j = j[k_o/\omega\mu_o]\tilde{K}_j \cdot \nabla\pi_j \quad (3-23.6)$$

where (\tilde{K}_j) is

$$(\tilde{K}_j) = \begin{bmatrix} n_p c_{2j} & -n_{tj} c_{3j} + n_p c_{1j} & 0 \\ n_{tj} c_{3j} - n_p c_{1j} & n_p c_{2j} & 0 \\ 0 & 0 & -(n_{tj}^2/n_p) c_{2j} \end{bmatrix} \quad (3-23.7)$$

From these equations one can see that the expressions for E_{zj} and H_{zj} are given as

$$E_{zj} = c_{3j} k_o n_{tj} \pi_j \quad (3-24.1)$$

$$H_{zj} = [1/(\omega\mu_o)] c_{2j} k_o^2 n_{tj}^2 \pi_j \quad (3-24.2)$$

This relation shows that in general E_z and H_z exist together.

From Eq. (3-23) one can derive an expression for the convection current \tilde{I} using the relation

$$\text{Curl } \tilde{H} - j\omega\epsilon_o \tilde{E} = \tilde{I} \quad (3-25)$$

The convection current carried by the electrons then can be given as

$$\tilde{I}_j = \omega\epsilon_o \tilde{V}_j \cdot \nabla\pi_j \quad (3-26.1)$$

where (V_j) is given as

$$(V_j) = \begin{bmatrix} -n_{tj}n_p c_{3j} + (n_p^2 - 1)c_{1j} & (-n_j^2 + 1)c_{2j} & 0 \\ (n_j^2 - 1)c_{2j} & -n_{tj}n_p c_{3j} + (n_p^2 - 1)c_{1j} & 0 \\ 0 & 0 & (n_{tj}/n_p)(-n_{tj}^2 + 1)c_{3j} - n_{tj}^2 c_{1j} \end{bmatrix} \quad (3-26.2)$$

To complete the discussion, the expression for the ac N is given by

$$N_j = -j(\epsilon_0 k_o^2 / e) n_{tj} (n_{tj} c_{1j} + n_p c_{3j}) \pi_j. \quad (3-26.3)$$

From the approximate roots of the warm plasma dispersion equation obtained in the last chapter, one can conclude that, as long as any one of the inequalities (2-46.1) and (2-46.2) together with the inequality

$$|WF_j| \ll 1 \quad (3-27)$$

is satisfied, the expressions for C_{ij} ($i = 1, 2, 3; j = 1, 2$) in Eqs. (2-22) will be very close to those found for the cold plasma case. In other words, under this condition, two of the characteristic waves become almost identical to the ordinary and extraordinary waves of cold plasma, and their normalized value E_{ij}/C_{oj} (for $i = r, \phi, z; j = 1, 2$) will not

be very large. For the third characteristic wave, i.e. the plasma wave, however, E_{i3}/C_{o3} (for $i = r, \phi, z$) assumes very large values. Because

$$|n_{tj}| \ll |n_{t3}| \quad j = 1, 2$$

implies that

$$|c_{ij}| \ll |c_{i3}| \quad i = 1, 2, 3; \quad j = 1, 2.$$

For the plasma wave the following approximate expressions can be given:

$$c_{13} \approx k_o^4 n_p [(1-x-y^2)/w]^{3/2}$$

$$c_{23} \approx j k_o^4 y n_p [(1-x-y^2)/w]^{1/2}$$

$$c_{33} \approx k_o^4 n_p^2 [(1-x-y^2)/w].$$

Under the condition of inequalities (2-46.1), (2-46.2) and 3-27) for

$$n_p \ll w^2 \quad (3-28.1)$$

one has

$$n_{t1} \approx [(-1+2x-x^2+y^2)/(-1+x+y^2)]^{1/2} + o(w^2) \quad (3-28.2)$$

$$n_{t2} \approx (1-x)^{1/2} + o(w^2) \quad (3-28.3)$$

$$n_{t3} \approx [(1-x-y^2)/w]^{1/2} + o(w^{1.5}) \quad (3-28.4)$$

$$c_{11,2} = k_o^4 o(n_p) \quad (3-28.5)$$

$$c_{13} = k_o^4 o(n_p) o(w^{-1.5}) \quad (3-28.6)$$

$$c_{21,2} = k_o^4 o(n_p) \quad (3-28.7)$$

$$c_{23} = k_o^4 o(n_p) o(w^{-.5}) \quad (3-28.8)$$

$$c_{31,2} = k_o^4 o(1) \quad (3-28.9)$$

$$c_{33} = k_o^4 o(w^{-1}) \quad (3-28.10)$$

and hence,

$$\begin{bmatrix} E_r & E_\phi & E_z \\ H_r & H_\phi & H_z \\ I_r & I_\phi & I_z \end{bmatrix}_{1,2} = \begin{bmatrix} o(n_p) & o(n_p) & o(1) \\ o(n_p^2) + o(1) & o(1) & o(n_p) \\ o(n_p) & o(n_p) & o(1) \end{bmatrix} \quad (3-28.11)$$

$$N_{1,2} = o(n_p) \quad (3-28.12)$$

$$\begin{bmatrix} E_r & E_\varphi & E_z \\ H_r & H_\varphi & H_z \\ I_r & I_\varphi & I_z \end{bmatrix}_3 = \begin{bmatrix} 0(n_p)0(W^{-2}) & 0(n_p)0(W^{-1}) & 0(W^{-1.5}) \\ & +m0(n_p)0(W^{-1.5}) & \\ 0(n_p^2)0(W^{-1}) & 0(W^{-2}) & 0(n_p)0(W^{-1.5}) \\ & +m0(W^{-1.5}) & \\ 0(n_p)0(W^{-2}) & 0(n_p)0(W^{-2}) & 0(W^{-2.5}) \end{bmatrix} \quad (3-28.13)$$

$$N_3 = 0(n_p)0(W^{-2.5}). \quad (3-28.14)$$

From Eqs. (3-28) one can see that for $n_p = 0$ all three waves become TM, having E_z , H_t and I_z components only. Remembering that $W \ll 1$, from Eqs. (3-28.14) it is also seen that, beyond the cut-off, the third wave is mainly a TEM wave with the components E_r , H_φ and I_z .

3.4 Derivation of the Characteristic Equation of the Waveguide Filled with Warm Plasma

The boundary conditions require that the tangential electric field vanishes on the waveguide wall. In addition, it is also assumed that on the boundary the normal component of electron velocity vanishes, so that no convection current flows into the guide wall and the neutrality of the plasma is maintained. To summarize, the boundary conditions are

$$\vec{E}_t(P) \times \vec{n} = 0 \quad (3-29.1)$$

$$\underline{E}_z(P) = 0 \quad (3-29.2)$$

$$\underline{I}(P) \cdot \underline{\hat{n}} = 0 \quad (3-29.3)$$

where P represents a point on the waveguide wall and $\underline{\hat{n}}$ is the unit vector normal to the guide wall at P.

As discussed in the last chapter, in a warm plasma there are three waves each associated with a refractive index surface, namely the ordinary, extraordinary and plasma waves. The total wave in the waveguide then can be expressed as

$$\underline{E} = \delta_1 \underline{E}_1 + \delta_2 \underline{E}_2 + \delta_3 \underline{E}_3 \quad (3-30.1)$$

$$\underline{H} = \delta_1 \underline{H}_1 + \delta_2 \underline{H}_2 + \delta_3 \underline{H}_3 \quad (3-30.2)$$

$$\underline{I} = \delta_1 \underline{I}_1 + \delta_2 \underline{I}_2 + \delta_3 \underline{I}_3 \quad (3-30.3)$$

$$\underline{N} = \delta_1 \underline{N}_1 + \delta_2 \underline{N}_2 + \delta_3 \underline{N}_3 \quad (3-20.4)$$

As a consequence of the boundary conditions (3-29) in general all these waves become coupled*. Inserting Eqs. (3-23) and (3-26) into Eqs. (3-30) making use of (3-29) one finds three linear homogeneous

*For $n_p = 0$, as it can be seen from Eqs. (3-28) the three waves can satisfy the boundary conditions separately and thus become uncoupled. Unlike the waves in cold plasma, three of them are TM waves having their electric field parallel to \underline{B}_0 . Therefore, simple modal solutions of zeroth order can be found even for rectangular waveguides with \underline{B}_0 perpendicular to one of the guide walls.

equations for δ_1 , δ_2 , and δ_3 . In order to have a nonzero solution for δ_i 's, the determinant of the set of equations must be zero.

The resulting characteristic equation for a circular waveguide is very complicated and can be expressed as follows:

$$\begin{vmatrix} \Phi_1 & \Phi_2 & \Phi_3 \\ Z_1 & Z_2 & Z_3 \\ R_1 & R_2 & R_3 \end{vmatrix} = 0 \quad (3-31.1)$$

where

$$\Phi_j = [m/(k_o r_o)] [(-C_{2j}^i + C_{1j}^i) J_m(n_{tj} k_o r_o) + C_{2j}^i n_{tj} J_{m+1}(n_{tj} k_o r_o)] \quad (3-31.2)$$

$$Z_j = C_{3j}^i J_m(n_{tj} k_o r_o) \quad (3-31.3)$$

$$\begin{aligned} R_j = & [m/(k_o r_o)] [(-n_p^2 + 1 - n_t^2) C_{2j}^i + (n_p^2 - 1) C_{1j}^i - C_{3j}^i] J_m(n_{tj} k_o r_o) \\ & - [(n_p^2 - 1) C_{1j}^i - C_{3j}^i] n_{tj} J_{m+1}(n_{tj} k_o r_o) \end{aligned} \quad (3-31.4)$$

$$\begin{aligned} C_{1j}^i = & F_j - 1 + (n_{tj}^2 + n_p^2) + W F_j [1 - F_j - 2(n_{tj}^2 + n_p^2) + Y^2 F_j] \\ & + W^2 F_j^2 (n_{tj}^2 + n_p^2 - Y^2 n_p^2) \end{aligned} \quad (3-31.5)$$

$$C_{2j}^1 = -YF_j(1-W) \quad (3-31.6)$$

$$C_{3j}^1 = 1 - n_{tj}^2 - 2n_p^2 + (n_{tj}^2 + n_p^2)n_p^2 + F_j(-2 + n_{tj}^2 + 2n_p^2 + X) \\ + WF_j[n_{tj}^2 - 2(n_{tj}^2 + n_p^2)n_p^2 - Xn_p^2] \quad (3-31.7)$$

$$n_{tj} = \sqrt{\frac{G_2}{W} + 2\sqrt{\frac{P}{3}} \cos \left\{ \frac{1}{3} \left[(j-1)2\pi + \tan^{-1} \frac{\sqrt{-(q^2/4) - (p^3/27)}}{-q/2} \right] \right\}} \quad (3-31.8)$$

$$0 \leq \tan^{-1} \frac{\sqrt{-(q^2/4) - (p^3/27)}}{q/2} \leq \pi \quad (3-31.9)$$

$$G_2 = [-1 + X + Y^2] + W[-2 + 2X + (3 - Y^2)n_p^2] \quad (3-31.10)$$

$$G_1 = [(2 - 4X + 2X^2 + XY^2 - 2Y^2) + (-2 + 2X - XY^2 + 2Y^2)n_p^2] \\ + W[(1 - 2X + X^2) + (-4 + 4X + 2Y^2)n_p^2 + (3 - 2Y^2)n_p^4] \quad (3-31.11)$$

$$G_0 = [(-1 + 3X - 3X^2 + X^3 - XY^2 + Y^2) + (2 - 4X + 2X^2 + 2XY^2 - 2Y^2)n_p^2 \\ + (-1 + X - XY^2 + Y^2)n_p^4] + W[(1 - 2X + X^2 - Y^2)n_p^2 + (-2 + 2X + 2Y^2)n_p^4 \\ + (1 - Y^2)n_p^6] \quad (3-31.12)$$

$$-p/3 = (G_2/3W)^2 - G_1/(3W) \quad (3-31.13)$$

$$-q/2 = -(G_2/3W)^3 + G_1 G_2/(6W^2) - G_0/(2W) \quad (3-31.14)$$

The quantities used in the above equations are related in the following manner:

$$c_{1j}^i = c_1/(k_o^4 n_p n_{tj}) \quad (3-32.1)$$

$$c_{2j}^i = c_2/(-j k_o^4 n_p n_{tj}) \quad (3-32.2)$$

$$c_{3j}^i = c_3/k_o^4 \quad (3-32.3)$$

$$\Phi_j = E_{\varphi j}(r_o)/(k_o^5 n_p n_{tj}) \quad (3-32.4)$$

$$Z_j = -E_{zj}(r_o)/(k_o^5 n_{tj}) \quad (3-32.5)$$

$$R_j = I_{rj}(r_o)/(\omega_o k_o^5 n_p n_{tj}) \quad (3-32.6)$$

The above quantities are introduced in order to make the characteristic equation dimensionless.

For $j = 3$, we have the following approximations if $W \ll 1$:

$$n_{t3} = \alpha^{-.5} \quad (3-33.1)$$

$$C'_{13} \cong \alpha^{-1} \quad (3-33.2)$$

$$C'_{23} \cong -\gamma \quad (3-33.3)$$

$$C'_{33} \cong \alpha^{-1} n_p^2 \quad (3-33.4)$$

$$\text{where } \alpha = W/(1-X-\gamma^2). \quad (3-33.5)$$

Hence, Φ_3 , Z_3 and R_3 can be approximated by

$$\Phi_3 = \lambda_m \alpha^{-1} J_m(\alpha^{-0.5} R) - \gamma \alpha^{-0.5} J_{m+1}(\alpha^{-0.5} R) \quad (3-33.6)$$

$$Z_3 = -\alpha^{-1} n_p^2 J_m(\alpha^{-0.5} R) \quad (3-33.7)$$

$$R_3 = \lambda_m \alpha^{-1} (\gamma - 1) J_m(\alpha^{-0.5} R) + \alpha^{-1.5} J_m(\alpha^{-0.5} R) \quad (3-33.8)$$

where

$$\lambda_m = m/(k_o r_o) \quad (3-33.9)$$

$$R = k_o r_o \quad (3-33.10)$$

Considering the case that n_{t1} and n_{t2} are much smaller than n_{t3} , Eqs. (3-31) and (3-33) reveal that generally

$$|\Phi_1|, |\Phi_2| \ll |\Phi_3| \quad (3-34.1)$$

$$|Z_1|, |Z_2| \ll |Z_3| \quad (3-34.2)$$

$$|R_1|, |R_2| \ll |R_3| \quad (3-34.3)$$

and also

$$|\Phi_3|, |Z_3| \ll |R_3|. \quad (3-34.4)$$

Expanding Eq. (3-31.1) and making use of Eqs. (3-33) one obtains the following equation:

$$\begin{aligned} & [\lambda_m \alpha^{0.5} J_m(\alpha^{-0.5} R) - \gamma \alpha J_{m+1}(\alpha^{-0.5} R)] (Z_1 R_2 - Z_2 R_1) \\ & + \alpha^{0.5} n_p^2 J_m(\alpha^{-0.5} R) (\Phi_2 R_1 - \Phi_1 R_2) \\ & + [\lambda_m \alpha^{0.5} (\gamma - 1) J_m(\alpha^{-0.5} R) + J_{m+1}(\alpha^{-0.5} R)] (\Phi_1 Z_2 - \Phi_2 Z_1) = 0 \quad (3-35) \end{aligned}$$

For the regions defined by inequalities (2-47.6), (2-47.7) and (3-27), α , as defined by Eq. (3-33.4) is very small; hence, in Eq. (3-35) the first two terms which are $O(\alpha^{0.5})$ can be neglected and the characteristic equation reduces to

$$J_{m+1}(\alpha^{0.5} R) [\Phi_1 Z_2 - \Phi_2 Z_1] + O(\alpha^{0.5}) = 0; \quad R = k_o r_o. \quad (3-36)$$

Thus neglecting the term $O(\alpha^{0.5})$, the roots n_p of the characteristic equation can be divided into the following kinds:

- 1) The roots of

$$\epsilon_1 Z_2 - \epsilon_2 Z_1 = 0. \quad (3-37)$$

This equation can be identified as the characteristic equation for the cold plasma case. Let us name the modes which correspond to these roots as "quasi-optical modes."

- 2) The roots which are determined by

$$J_{m+1}(\alpha^{-0.5} R) = 0; \quad R = k_o r_o. \quad (3-38)$$

It should be pointed out that in this approximate equation n_p does appear explicitly and this shows that n_p can assume almost any value as long as the approximation is valid. Thus in an n_p versus $k_o r_o$ diagram those roots will be shown by approximately vertical parallel lines. However, the exact roots which belong to this class slightly deviate from the approximate solution. As will be seen in Chapter V, the value of n_p is very sensitive to α or $k_o r_o$; i.e. a large variation in n_p may result from a small change in α or $k_o r_o$.

The modes which correspond to these roots may be called "plasma modes."

After having found n_p , $J_m(n_{tj}R)$'s can be computed; then the relative magnitudes of various waves for each mode can be solved from the set of equations:

$$\Phi_1 n_{t1} \delta_1 + \Phi_2 n_{t2} \delta_2 + \Phi_3 n_{t3} \delta_3 = 0 \quad (3-39.1)$$

$$Z_1 n_{t1} \delta_1 + Z_2 n_{t2} \delta_2 + Z_3 n_{t3} \delta_3 = 0 \quad (3-39.2)$$

$$R_1 n_{t1} \delta_1 + R_2 n_{t2} \delta_2 + R_3 n_{t3} \delta_3 = 0 \quad (3-39.3)$$

In the following we shall assume that inequalities (2-47.6), (2-47.7) and (3-27) are satisfied.

(a) For quasi optical modes, making use of Eqs. (3-39.1) and (3-39.2), (3-23) and (3-26) we obtain the relative magnitudes of three characteristic waves as in the following:

$$\delta_3 = \delta_{1,2}(\alpha^2) \quad (3-40.1)$$

$$\delta_3 \begin{bmatrix} E_r & E_\phi & E_z \\ H_r & H_\phi & H_z \\ I_r & I_\phi & I_z \end{bmatrix}_3 = \delta_{1,2} \begin{bmatrix} E_r 0(\alpha^0) & E_\phi 0(\alpha^{0.5}) & E_z 0(\alpha^{0.5}) \\ H_r 0(\alpha^{0.5}) & H_\phi 0(1) & H_z 0(\alpha^{0.5}) \\ I_r 0(1) & I_\phi 0(\alpha) & I_z 0(\alpha^{-0.5}) \end{bmatrix}_{1,2}$$

$$(3-40.2)$$

$$\delta_3 N_3 = \delta_{1,2} N_{1,2} O(\alpha^{-0.5}) \quad (3-40.3)$$

Using the above results we can compare the solutions of cold and compressible fluid plasma models and conclude the following: For both models the propagation constants are almost the same. E_ϕ , E_z , H_r and H_z are also unchanged. However, with the introduction of the compressibility in the plasma E_r and H_ϕ are modified with a considerable contribution from the plasma waves. The transverse component of the convection current I_r , has contributions from the optical waves as well as from the plasma wave. The axial component of the convection current and the density, however, are essentially due to the plasma wave.

(b) For the plasma modes, Eqs. (3-39), (3-23) and (3-26) yield the following results:

$$\delta_3 = \delta_{1,2} O(\alpha^{1.5}) \quad (3-41.1)$$

$$\delta_3 \begin{bmatrix} E_r & E_\phi & E_z \\ H_r & H_\phi & H_z \\ I_r & I_\phi & I_z \end{bmatrix}_3 = \delta_{1,2} \begin{bmatrix} E_r O(\alpha^{-0.5}) & E_\phi O(1) & E_z O(1) \\ H_r O(1) & H_\phi O(\alpha^{-0.5}) & H_z O(1) \\ I_r O(\alpha^{-0.5}) & I_\phi O(\alpha^{0.5}) & I_z O(\alpha^{-1}) \end{bmatrix}_{1,2} \quad (3-41.2)$$

$$\delta_3 N_3 = \delta_{1,2} N_{1,2} O(\alpha^{-1}) \quad (3-41.3)$$

Earlier in the discussion of Eq. (3-38) we have shown that the propagation constant n_p of these modes are very sensitive to the transverse propagation constant of the plasma wave or the temperature. The above results, however, show that the field components of those modes are contributed essentially by plasma waves. And these modes behave almost like TEM waves having E_r , H_ϕ and I_z as the main field components. For this reason, these waves are called plasma modes.

For a given waveguide with fixed $B_{\sim 0}$ the waves of frequencies which satisfy

$$J_{m+1} \left[\sqrt{[1 - (\omega_H^2/\omega^2) - (\omega_N^2/\omega^2)]/W} (\omega r_0/c) \right] + O(\alpha^{0.5}) = 0 \quad (3-43)$$

will belong to these modes. As mentioned earlier, in a plot of n_p versus frequency, the hybrid modes will be forming a set of lines which are almost parallel to the n_p axis. One can see that to excite one of these modes individually will be almost impossible. Because, even if $B_{\sim 0}$ and the electron density could be maintained absolutely constant (which is an impossible condition), a very slight change of

frequency would cause a propagating mode to become an attenuating mode.

To summarize, the introduction of compressibility in the plasma brings about new modes as well as some modifications on the fields, despite that the k - β diagram of a cold plasma model retains strongly its own identity even for relatively high temperature. It is most interesting to find that , these new modes have propagation constants so densely packed that they become almost a continuous band. Because of the strong modification of the fields, in general, it is expected that the impedance of an antenna inside the guide may assume different values, depending on the plasma models.

IV. ORTHOGONALITY PROPERTIES AND POWER RELATIONS

Let us consider a medium of uniform anisotropic lossless plasma and confine our discussion to electrons only; let us suppose that in the medium we have an electric current source $\underline{j}_a(r)$, a magnetic current source $\underline{k}_a(r)$, a particle source which creates $d_t \rho$ particles per unit volume per second and a mechanical energy source with a mean force $\underline{F}_a(r)$ over all particles. Now we consider a second medium which is identical to the first except that the static magnetic field is reversed in direction. To distinguish source and field quantities in the two media we shall use the subscript "a" for one medium and "b" for the other. In the following we shall develop a relation for all field and source quantities in these two media.

From the Maxwell equations and the first two moment equations for both cases as stated above, one obtains:

$$\begin{aligned}
 & \iint_S \{ [\underline{E}_a \times \underline{k}_b - T_e (N_a/N_o) \underline{L}_b] - [\underline{E}_b \times \underline{k}_a - T_e (N_b/N_o) (\underline{L}_a)] \} \cdot d\underline{S} \\
 &= \iiint_V \{ [\underline{E}_b \cdot \underline{j}_a - \underline{k}_b \cdot \underline{k}_a + e T_e N_b d_t \rho_a - (1/e) \underline{L}_b \cdot \underline{F}_a] \\
 &- [\underline{E}_a \cdot \underline{j}_b - \underline{k}_a \cdot \underline{k}_b + e T_e N_a d_t \rho_b - (1/e) \underline{L}_a \cdot \underline{F}_b] \} dV \quad (4-1.1)
 \end{aligned}$$

where $T_e = \hbar k T / e$, (4-1.2)

S is the closed surface of a volume V ; N_a and N_b denote the perturbed electron density in media "a" and "b", respectively, N_0 the average density which is assumed to be the same in both media; \hat{n} is the outward normal of S .

If one writes the Maxwell equations and the first two moment equations in ω domain for the medium "a" and then takes the complex conjugates of all four equations one obtains the field equations for the medium "b." Thus the fields and sources in one medium are related to those in the other one by the following equations:

$$-E_a^* = E_b \quad (4-2.1)$$

$$H_a^* = H_b \quad (4-2.2)$$

$$-N_a^* = N_b \quad (4-2.3)$$

$$I_a^* = I_b \quad (4-2.4)$$

$$-F_a^* = F_b \quad (4-2.5)$$

$$\rho_a^* = \rho_b \quad (4-2.6)$$

$$-K_a^* = K_b \quad (4-2.7)$$

$$J_a^* = J_b \quad (4-2.8)$$

These results hold even if the medium contains perfect conductors, since in both cases it is required that $\vec{E} \times \vec{n} = 0$ and $\vec{I} \cdot \vec{n} = 0$ on the conductor.

Let us consider as before that \vec{B}_0 is parallel to the z axis and the guide axis parallel to the z' axis. Then the quantities in medium "a" can be written as

$$\vec{E}_a = \vec{e}_a \exp(\gamma_a' z') \quad (4-3.1)$$

$$\vec{H}_a = \vec{h}_a \exp(\gamma_a' z') \quad (4-3.2)$$

$$\vec{I}_a = \vec{i}_a \exp(\gamma_a' z') \quad (4-3.3)$$

$$\vec{N}_a = \vec{n}_a \exp(\gamma_a' z') \quad (4-3.4)$$

where γ_a' is the constant along z' the guide axis as distinguished from γ which is the propagation constant along z or \vec{B}_0 . These expressions will be used later.

Since both (4-1) and (4-2) hold for any two fields as long as they are solutions to the Maxwell and transport equations for media "a" and "b" respectively, one can obtain the following general relation between two fields subindexed by "m" and "n" respectively in a source free region enclosed by a surface S:

$$\iint_S \{ [\tilde{E}_{am} \times \tilde{H}_{an}^* - T_e (N_{am}/N_o) \tilde{I}_{an}^*] - [(-\tilde{E}_{an}^*) \times \tilde{H}_{am} - T_e (-N_{an}^*/N_o) \tilde{I}_{am}] \} \cdot d\tilde{S} = 0 \quad (4-4)$$

Now this relation hold for any field quantities in the same medium; therefore, for simplicity the subscript "a" may be dropped in the following discussion.

Let S be composed of the following three surfaces : S_o and S_1 perpendicular to the guide axis at $z^1 = 0$ and $z^1 = z_o^1$ respectively and the third S_2 coinciding with the waveguide walls as shown in Figure (4-1). Since on S_2 , \tilde{E}_m and \tilde{E}_n^* have vanishing tangential components and \tilde{I}_m and \tilde{I}_n^* have vanishing normal components Eq. (4-4) becomes

$$\begin{aligned} & \iint_{S_o} \{ [\tilde{E}_m \times \tilde{H}_n^* - T_e (N_m/N_o) \tilde{I}_n^*] \\ & + [\tilde{E}_n^* \times \tilde{H}_m - T_e (N_n^*/N_o) \tilde{I}_m] \} \cdot \hat{z}^1 dS \\ & = \iint_{S_1} \{ [\tilde{E}_m \times \tilde{H}_n^* - T_e (N_m/N_o) \tilde{I}_n^*] \\ & + [\tilde{E}_n^* \times \tilde{H}_m - T_e (N_n^*/N_o) \tilde{I}_m] \} \cdot \hat{z}^1 dS \end{aligned} \quad (4-5)$$

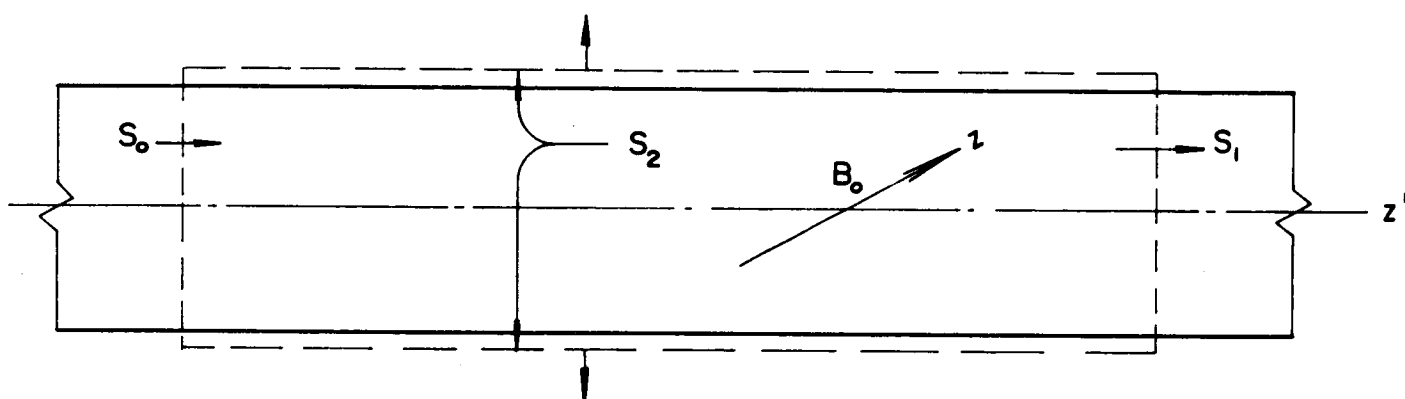


Figure (4-1) Longitudinal cross section of a waveguide

or using the relations (4-3)

$$\begin{aligned}
 & \iint_{S_0} \{ [\tilde{e}_m \times \tilde{h}_n^* - T_e(n_m/N_0) \tilde{i}_n^*] + [\tilde{e}_n^* \times \tilde{h}_m - T_e(n_n^*/N_0) \tilde{i}_m] \} \cdot \hat{z}' dS \\
 &= e^{(\gamma_m^* + \gamma_n^*) z_0'} \iint_{S_1} \{ [\tilde{e}_m \times \tilde{h}_n^* - T_e(n_m/N_0) \tilde{i}_n^*] \\
 & \quad + [\tilde{e}_n^* \times \tilde{h}_m - T_e(n_n^*/N_0) \tilde{i}_m] \} \cdot \hat{z}' dS. \quad (4-6)
 \end{aligned}$$

Since z_0' is arbitrary, one has

$$\iint_S \{ [\tilde{e}_m \times \tilde{h}_n^* - T_e(n_m/N_0) \tilde{i}_n^*] + [\tilde{e}_n^* \times \tilde{h}_m - T_e(n_n^*/N_0) \tilde{i}_m] \} \cdot d\tilde{S} = 0 \quad (4-7.1)$$

$$\text{if } \gamma_m^* \neq -\gamma_n^* \quad (4-7.2)$$

where s is any cross-section area of the guide. Actually this result is valid even if the static magnetic field is not parallel to the guide axis.

From the Maxwell and the moment equations it can be seen that if the quantities with subscripts "m" as indicated below on the left side column satisfy the equations, then the corresponding quantities listed on the right side column also satisfy the same equations:

$$(\partial/\partial z)_m \rightarrow -(\partial/\partial z)_m \text{ or } \gamma_m \rightarrow -\gamma_m \quad (4-8.1)$$

$$\dot{\mathcal{E}}_{tm} \rightarrow -\dot{\mathcal{E}}_{tm} \quad (4-8.2)$$

$$\mathcal{H}_{tm} \rightarrow \mathcal{H}_{tm} \quad (4-8.3)$$

$$\dot{\mathcal{E}}_{zm} \rightarrow \dot{\mathcal{E}}_{zm} \quad (4-8.4)$$

$$\mathcal{H}_{zm} \rightarrow -\mathcal{H}_{zm} \quad (4-8.5)$$

$$\mathcal{L}_{tm} \rightarrow -\mathcal{L}_{tm} \quad (4-8.6)$$

$$\mathcal{L}_{zm} \rightarrow \mathcal{L}_{zm} \quad (4-8.7)$$

$$\mathcal{K}_m \rightarrow -\mathcal{K}_m \quad (4-8.8)$$

$$\mathcal{F}_{tm} \rightarrow -\mathcal{F}_{tm} \quad (4-8.9)$$

$$\mathcal{F}_{zm} \rightarrow \mathcal{F}_{zm} \quad (4-8.10)$$

$$\rho_m \rightarrow -\rho_m \quad (4-8.11)$$

$$\chi_{zm} \rightarrow -\chi_{zm} \quad (4-8.12)$$

$$\ell_{tm} \rightarrow -\ell_{tm} \quad (4-8.13)$$

$$\ell_{zm} \rightarrow \ell_{zm} \quad (4-8.14)$$

Now since all the quantities listed on the right side column satisfy the equations, one can substitute them in Eq. (4-6) for those quantities subindexed 'm' and obtain the following relation if $\vec{z}' // \vec{z}$ or \vec{B}_0 is along the guide axis.

$$\iint_S \{ -[\vec{e}_m \times \vec{h}_n^* - T_e (n_m/N_o) \vec{i}_n^*] + [\vec{e}_n^* \times \vec{h}_m - T_e (n_n^*/N_o) \vec{i}_m] \} \{ 1 - \exp[(-\gamma_m + \gamma_n^*) z_o] \} \cdot d\vec{S} = 0 \quad (4-9)$$

Thus, if

$$\vec{B}_0 // \vec{z}' \text{ (or } \vec{z} // \vec{z}'), \quad (4-10.1)$$

$$\text{and } \gamma_m \neq \gamma_n^*, \quad (4-10.2)$$

then one has

$$\begin{aligned} & \iint_S \{ -[\vec{e}_m \times \vec{h}_n^* - T_e (n_m/N_o) \vec{i}_n^*] \\ & + [\vec{e}_n^* \times \vec{h}_m - T_e (n_n^*/N_o) \vec{i}_m] \} \cdot d\vec{S} = 0 \end{aligned} \quad (4-10.3)$$

where s is any cross-section area of the guide.

By combining Eqs. (4-7) and (4-10) one can summarize the following results for a waveguide with a fixed static magnetization:

Case 1) If \vec{B}_0 is not necessarily parallel to z^1 , the guide axis, and $\pm \gamma_m^1 \neq \gamma_n^{1*}$ (4-11.1)

then

$$\iint_s \{ [\vec{e}_m \times \vec{h}_n^* - T_e(n_m/N_o) \vec{i}_n^*] + [\vec{e}_n^* \times \vec{h}_m - T_e(n_n^*/N_o) \vec{i}_m] \} \cdot d\vec{S} = 0. \quad (4-11.2)$$

If in addition to the condition

$$\pm \gamma_m \neq \gamma_n^* \quad (4-12.1)$$

also $\vec{B}_0 // z^1$ (the guide axis) (4-12.2)

$$\iint_s [\vec{e}_m \times \vec{h}_n^* - T_e(n_m/N_o) \vec{i}_n^*] \cdot d\vec{S} = 0. \quad (4-12.3)$$

Case 2) If

$$-\gamma_m^1 = \gamma_n^{1*} \text{ but } \gamma_m^1 \neq \gamma_n^{1*} \quad (4-13.1)$$

and

$$\vec{B}_0 // z^1 \quad (4-13.2)$$

then

$$\begin{aligned} & \iint_s \{ -[\vec{e}_m \times \vec{h}_n^* - T_e(n_m/N_o) \vec{i}_n^*] \\ & + [\vec{e}_n^* \times \vec{h}_m - T_e(n_n^*/N_o) \vec{i}_m] \} \cdot d\vec{S} = 0. \end{aligned} \quad (4-13.3)$$

$$\text{Case 3) If } \gamma_m^{\perp} = \gamma_n^{\perp *} \text{ but } \gamma_m^{\perp} \neq -\gamma_n^{\perp *} \quad (4-14.1)$$

then

$$\iint_S \{ [\underline{e}_m \times \underline{h}_n^* - T_e(n_m/N_o) i^*] + [\underline{e}_n^* \times \underline{h}_m - T_e(n_n^*/N_o) i_m] \} \cdot d\vec{S} = 0. \quad (4-14.2)$$

which is also valid even if $\underline{B}_o \times \underline{z}^{\perp}$.

Using the above results one can determine the power carried by each mode in waveguides.

Let us first consider a field with complex propagation constant which is neither real nor purely imaginary. Then it is obvious that

$$\pm \gamma_m^{\perp} \neq \gamma_m^{\perp *} \quad (4-15.1)$$

Since the results obtained above are valid for any field m and n , in the following we consider the relation between the mode and itself. Then from Eq. (4-11.2) one obtains

$$\text{Re} \iint_S [\underline{e}_m \times \underline{h}_m^* - T_e(n_m/N_o) i^*] \cdot d\vec{S} = 0. \quad (4-15.2)$$

This result shows that fields with a propagation constant which is not real, carry no real power. This result is true in general, because Eq. (4-11.2) doesn't require that \underline{B}_o be parallel to \underline{z}^{\perp} . For a lossless guide this result is expected, because a complex propagation constant would imply attenuation of real power along the waveguide, in contradiction to the assumption of losslessness.

Next, for $\underline{B}_0 // \underline{z}$ (4-15.3.1)

from Eq. (4-12.3) one has

$$(\text{Re and Im}) \iint_S [\underline{e}_m \times \underline{h}_m^* - T_e (n_m / N_0) \underline{i}^*] \cdot d\underline{S} = 0. \quad (4-15.3.2)$$

Thus, with axial magnetization, modes with complex constant also carry no reactive power.

Let us assume that γ_m is real. Since now

$$\gamma_m = \gamma_m^* \quad (4-16.1)$$

Eq. (4-14.2) lead to the following relation

$$\text{Re} \iint_S [\underline{e}_m \times \underline{h}_m^* - T_e (n_m / N_0) \underline{i}^*] \cdot d\underline{S} = 0. \quad (4-16.2)$$

This is also generally true regardless of the direction of \underline{B}_0 for the same reason as before. Thus, as expected, the attenuating modes carry no real power flow.

Last, consider a propagation constant which is purely imaginary.

In this case

$$-\gamma_m = \gamma_m^*. \quad (4-17.1)$$

Then, from Eq. (4-14.3) one gets

$$\operatorname{Im} \int_S [\tilde{e}_m \times \tilde{h}_m^* - T_e (n_m/N_0) i^*] = 0 . \quad (4-17.2)$$

Thus the propagating modes carry no reactive power.*

In concluding this chapter, we may state that, as far as the real power is concerned, it is not necessary to consider the complex roots of the characteristic equation. If the dc magnetic field is parallel to the guide axis only the purely imaginary γ 's contribute to the real power. Therefore, in the next chapter only the real and purely imaginary roots of the characteristic equations of the wave guides will be considered.

*The energy relations can also be found in P. Allis, J. Buchsbaum and A. Bers "Waves in Anisotropic Plasmas"⁽³⁾ with a different approach of derivation.

V. SOLUTIONS OF CHARACTERISTIC EQUATION WHEN THE dc MAGNETIC FIELD IS PARALLEL TO THE GUIDE AXIS

It may be recalled that the characteristic equations derived in Chapter III for both the cold and the warm plasmas are even functions of the unknown n_p . Since, as concluded in the last chapter, we are interested only in the purely real or purely imaginary roots of n_p , it suffices to determine the real roots of n_p^2 in the characteristic equations, the positive roots of n_p^2 corresponding to propagating modes and the negative corresponding to attenuating modes.

5.1 Cold Plasma Case

In the case of cold plasma the characteristic equation of the waveguide can be written as

$$L J_m(n_{t1} k_o r_o) J_m(n_{t2} k_o r_o) + M J_{m+1}(n_{t1} k_o r_o) J_m(n_{t2} k_o r_o) + N J_m(n_{t1} k_o r_o) J_{m+1}(n_{t2} k_o r_o) = 0 \quad (5-1)$$

where

$$n_{t1,2} = [A(Bn_p^2 + C \pm U)]^{1/2}$$

$$L = [m/(v_1 - v_3^2)] U (v_1 - v_2) \{ (v_1 + v_2) [v_2^2 (v_1 + v_3) - v_1 v_2 v_3] + v_1^2 (v_3 - v_1)] n_p^2 - v_1 v_2 v_3 \}$$

$$M = -[v_2 R W_1 / (2v_3)] \{ (v_2^2 - v_1^2) (v_2^2 - v_1^2 + v_1 v_3) n_p^4$$

$$+ [v_2^2(2v_1+v_3)+2v_1^2(v_3-v_1)]n_p^2$$

$$+v_1(v_1-v_3)-U[(v_2^2-v_1^2)n_p^2+v_1]\}$$

$$N = [v_2^2 v_3 / (2v_3)] \{ (v_2^2 - v_1^2) (v_2^2 - v_1^2 + v_1 v_3) n_p^4$$

$$+ [v_2^2(2v_1+v_3)+2v_1^2(v_3-v_1)]n_p^2$$

$$+ v_1(v_1-v_3)+U[(v_2^2-v_1^2)n_p^2+v_1]\}$$

$$U = \{ (v_2^2 - v_1^2 + v_1 v_3)^2 n_p^4 + 2[v_2^2(v_1+v_3) - v_1(v_1-v_3)^2] n_p^2$$

$$+ (v_1-v_3)^2 \}^{1/2}$$

$$A = 1/(2v_1 v_3)$$

$$B = v_2^2 - v_1^2 - v_1 v_3$$

$$C = v_1 + v_3$$

$$v_1 = (1-\gamma^2-x)/(1-\gamma^2+x^2-2x)$$

$$v_2 = -x\gamma/(1-\gamma^2+x^2-2x)$$

$$v_3 = 1/(1-x)$$

$$R = k_o r_o$$

In this equation if X and Y are such that

$$Y = |1-X|$$

V_1 and V_2 and consequently U , n_{t1} , n_{t2} , L , M and N all become infinity for any value of n_p^2 . Thus, in such a case one cannot determine the roots. Also for the case where

$$Y = (1-X)^{1/2},$$

V_1 becomes zero and n_{t1} , n_{t2} and L become infinity and again one cannot find any roots of the characteristic equation. On the other hand as $X = 1$, V_3 goes to infinity and the characteristic equation can be simplified to the following form:

$$\begin{aligned} & -m(1-n_p^2)^{1/2}(1-1/Y)(1+1/Y^3)J_m[(1-n_p^2)^{1/2}k_0r_0] \\ & + (R/Y)[(-1+1/Y^2)n_p^4 + 2n_p^2 - 1]J_{m+1}[(1-n_p^2)^{1/2}k_0r_0] = 0. \end{aligned} \quad (5-2)$$

To find the real roots of n_p^2 in Eqs. (5-1) and (5-2), the regions of n_p^2 in which the left hand sides of these equations become real are studied. The lower and upper bounds of these regions are determined from one of the following three equations:

$$n_{t1} = n_{t2}$$

$$n_{t1} = 0$$

$$n_{t2} = 0$$

For any of these three cases the characteristic equation is always satisfied, regardless of the value of " $k_o r_o$ " and " m ." However, in these cases the total fields inside the waveguide are found to be zero also.

The regions of n_p^2 in which there might be a root of the characteristic equation depend only on " X " and " Y " and not on " m " and " $k_o r_o$." Figure (5-1) shows those regions for various values of X and Y .

From Figure (5-1) it is seen that the regions of n_p^2 in which a solution for the characteristic equation may exist can be at most two. The first region correspond to attenuating modes only, since its upper bound is always finite and negative and its lower bound is $-\infty$. Thus it can be called the region of attenuation. Most of the roots in the second region correspond to propagating modes except for a few cases where the attenuation constants are very small. The bounds of the second region are finite. Thus, it may generally be called the region of propagation. In some cases these two kinds of regions are connected forming a region of propagation and attenuation with a positive finite upper bound and a negative infinite lower bound. Regions of this type can be called connected regions of propagation and attenuation or simply connected regions.

As it is seen from Figure (5-1) for small X and Y , there are regions of which the upper bounds are higher than those of other cases. However, when X or Y or both assumes a large value, generally larger than 1, the propagating region ceases to exist; when X equals to 1,

the first and second regions become always connected. The determination of the lower and upper bounds of these regions is a matter of simple algebra, depending upon the characteristic equation. However, the terms involved have long expressions in X and Y .

In Figure (5-2) some plots of n_p versus $R = r_0 \omega / c$ are given for fixed values of X and Y . These curves show how the propagation constant n_p changes with the guide radius for various values of ω , ω_N^2 and ω_H . In general the loci of n_p become approximately parallel horizontal lines as the guide radius increases. However, for $0.8 \leq Y \leq 1$, the loci become quite irregular.

From these data one can also plot the familiar $k-\beta$ diagram or similarly ω vs n_p for fixed values of ω_N , ω_H and r_0 . An example for that is given in Figure (5-3). This diagram shows a set of propagating modes in a waveguide of radius r_0 with fixed values of the electron density and the dc magnetic field.

5.2 Warm Plasma Case

In the last section of Chapter III, the characteristic equation and its approximate solutions are derived, for plasma filled, axially magnetized waveguide, including the effect of the pressure. In this case, similar to the case of the cold plasma model, the real roots n_p^2 of the characteristic equation can take place only in certain intervals of n_p^2 . However, the algebra for the determination of the lower and upper bounds of these intervals involves the solution of polynomial of sixth degree, the coefficients of which are complicated functions

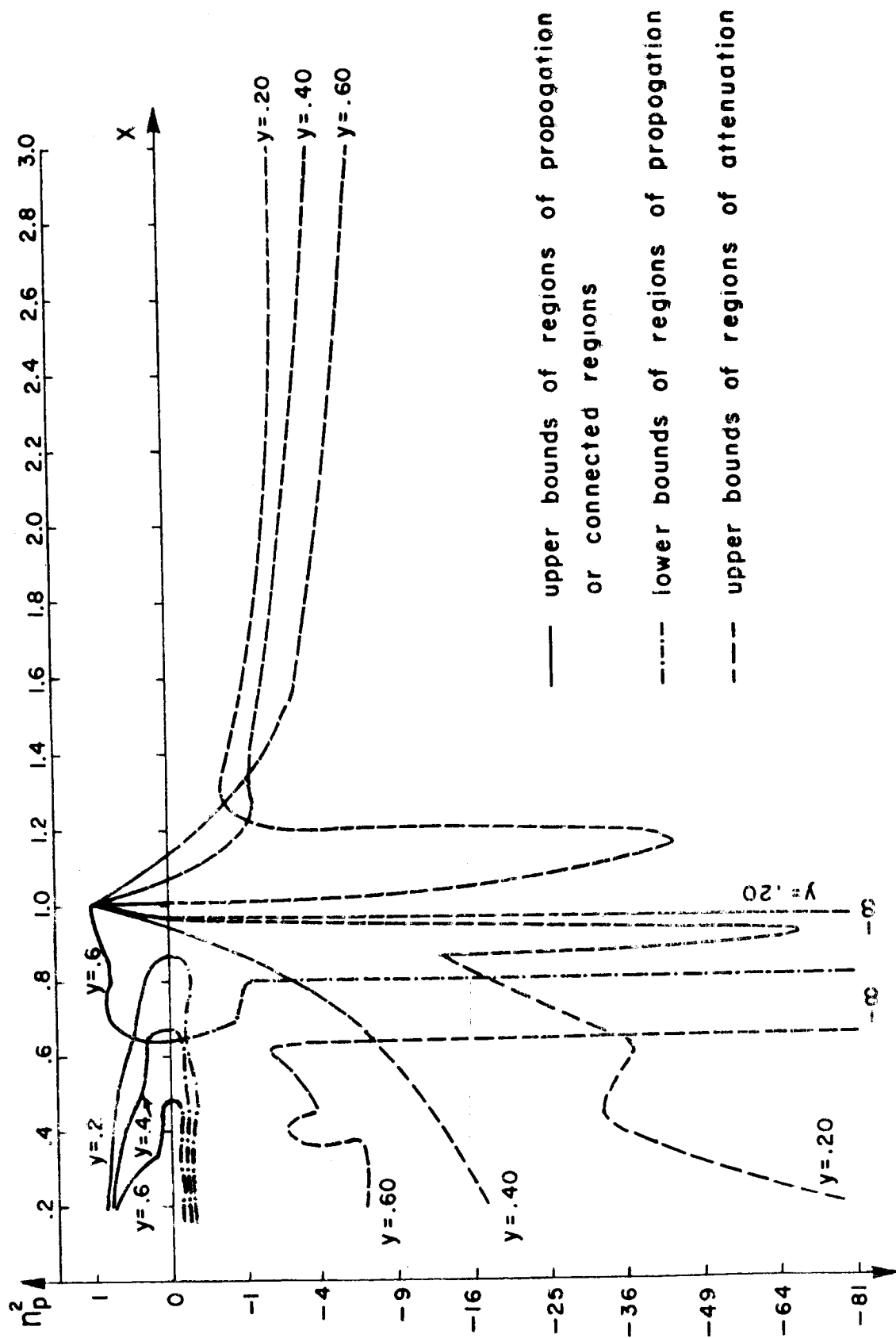


Figure (5-1) Regions of propagation in axially magnetized circular waveguides filled with cold plasma

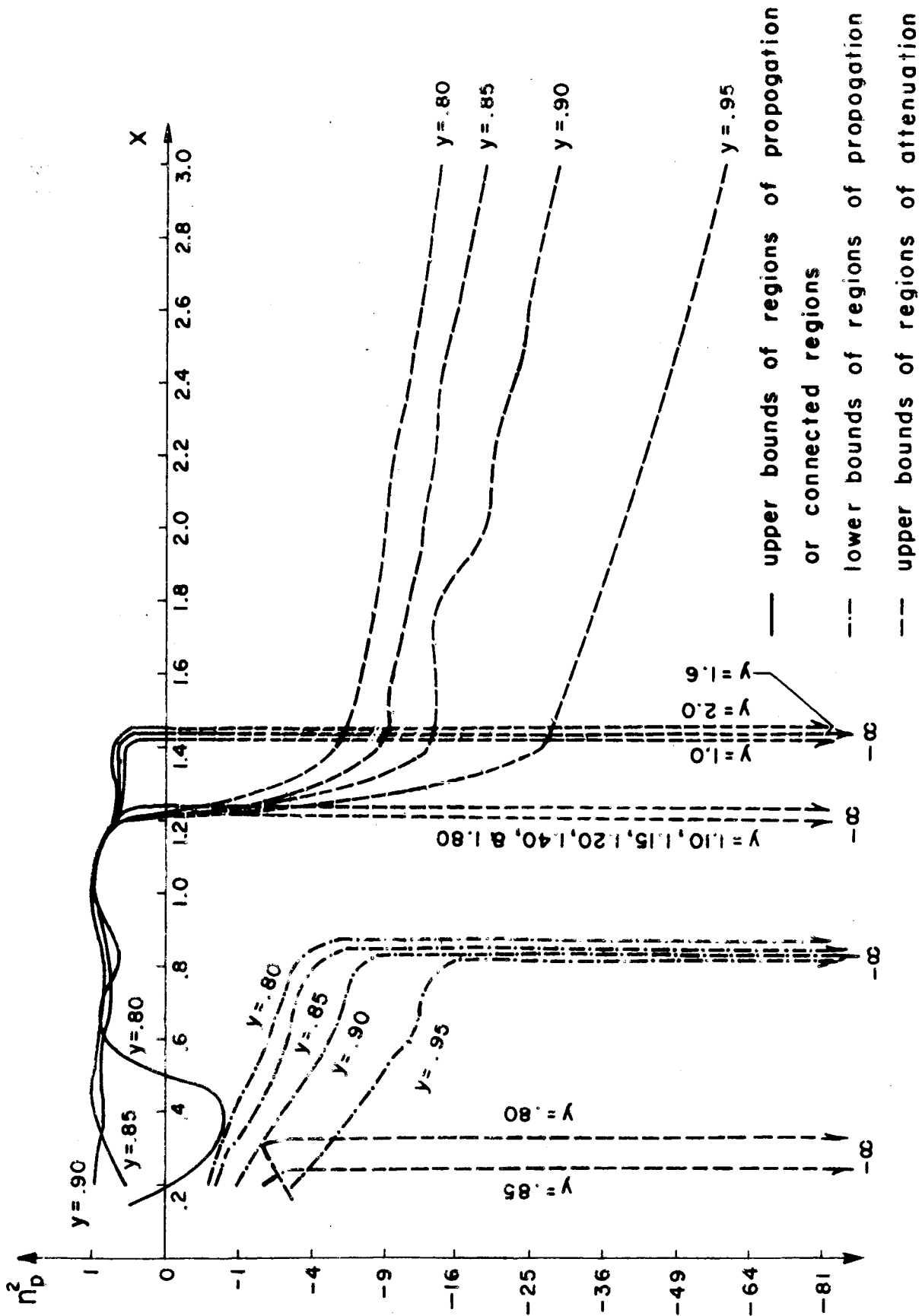


Figure (5-1) Continued

$X = .20$

$Y = .20$

--- atten.

--- boundaries of regions

— prop.

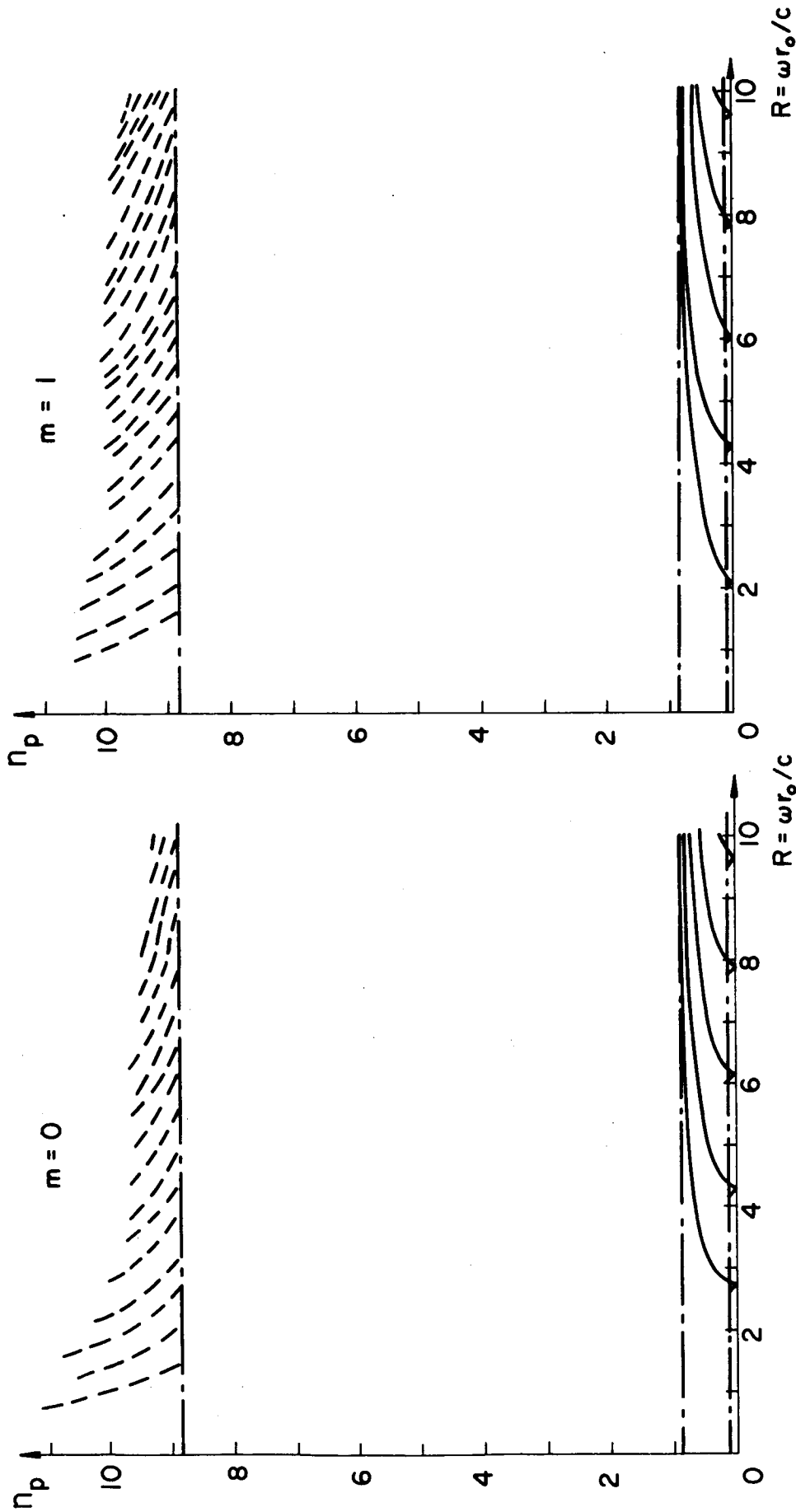


Figure (5-2) In a cold plasma filled guide axial propagation index as a function of the radius in free space wave length.

a) $Y = 0.2; 0.2 \leq X \leq 1.25$

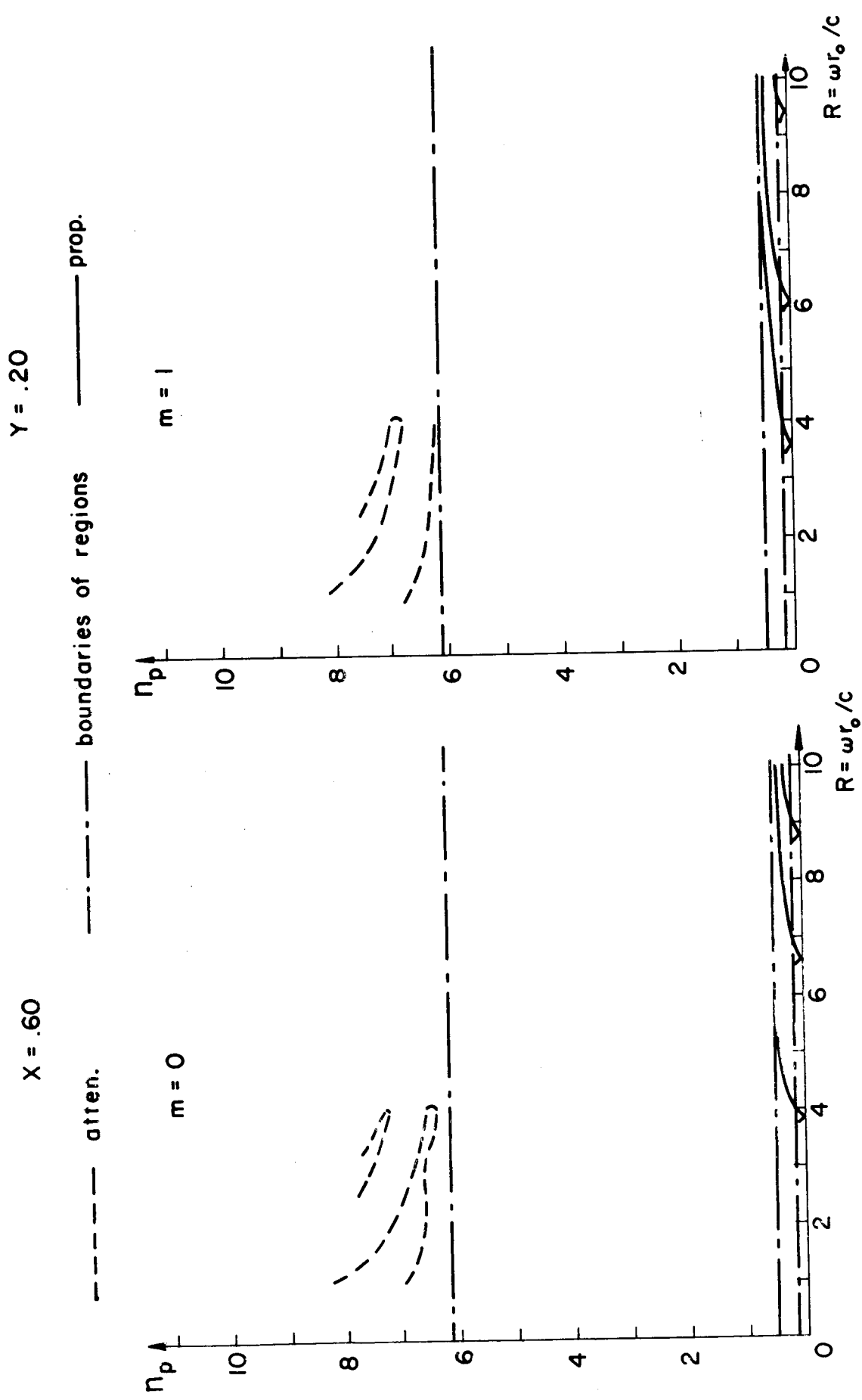


Figure (5-2) a) continued

$X = .85$

$Y = .20$

--- often. --- boundaries of regions --- prop.

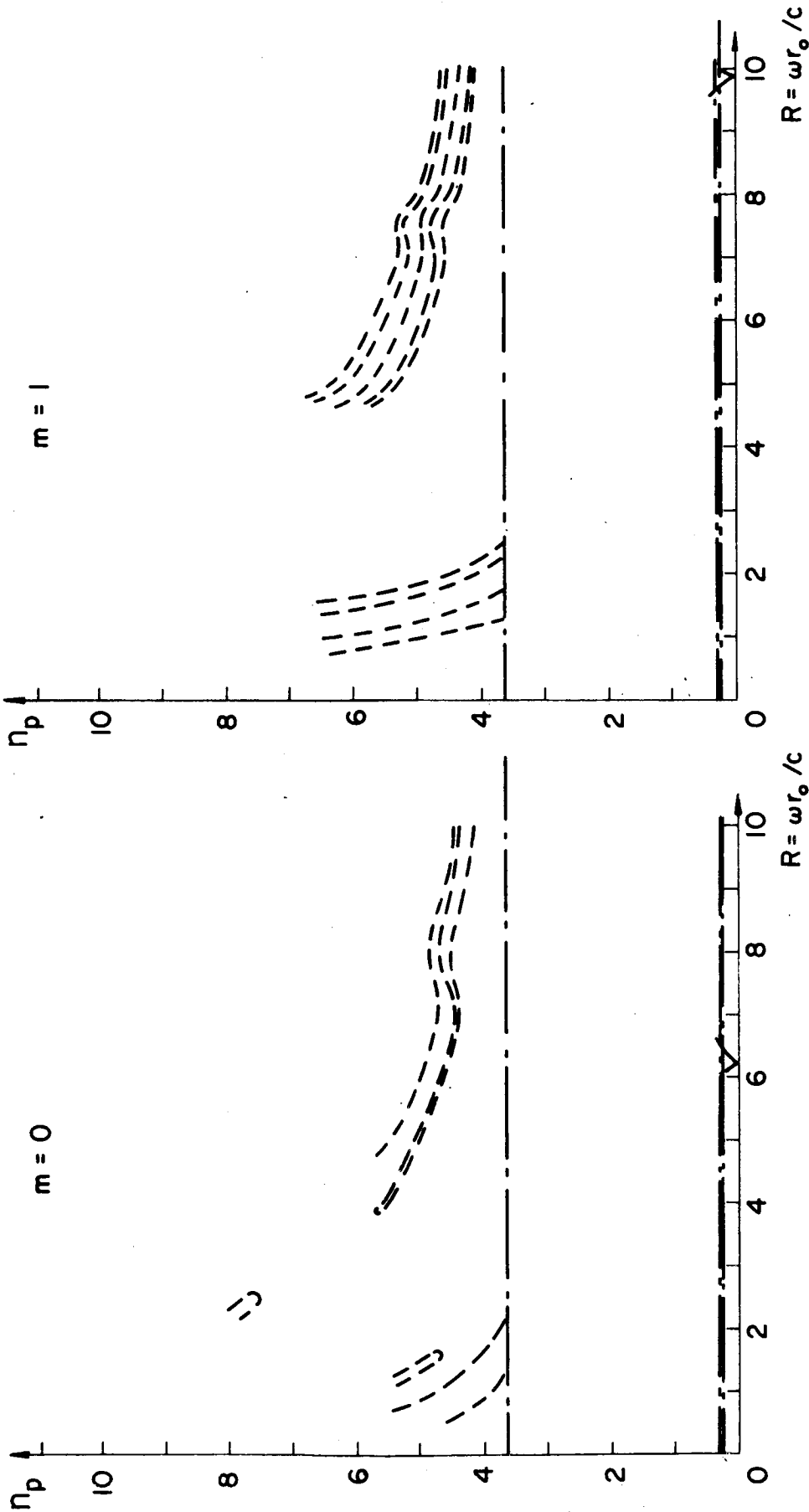


Figure (5-2) a) continued

$X = .91$

--- atten.

--- boundaries of regions

$Y = .20$

--- prop.

$m = 0$

$m = 1$

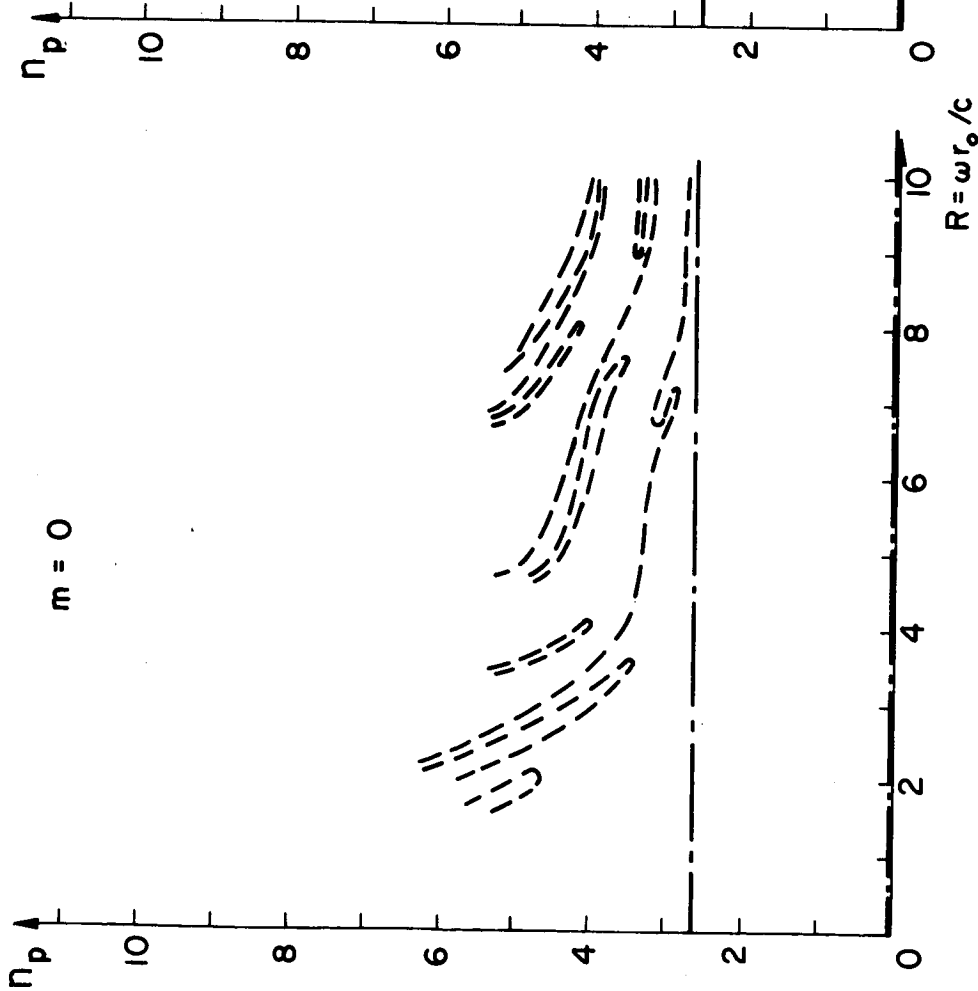


Figure (5-2) a) Continued

$X = 1.00$

$Y = .20$

----- atten.

----- boundaries of regions

----- prop.

n_p

n_p

$m = 0$

$m = 1$

10

10

8

8

6

6

4

4

2

2

0

0

2

4

6

8

10

$R = \omega r_0 / c$

$R = \omega r_0 / c$

Figure (5-2) a) Continued

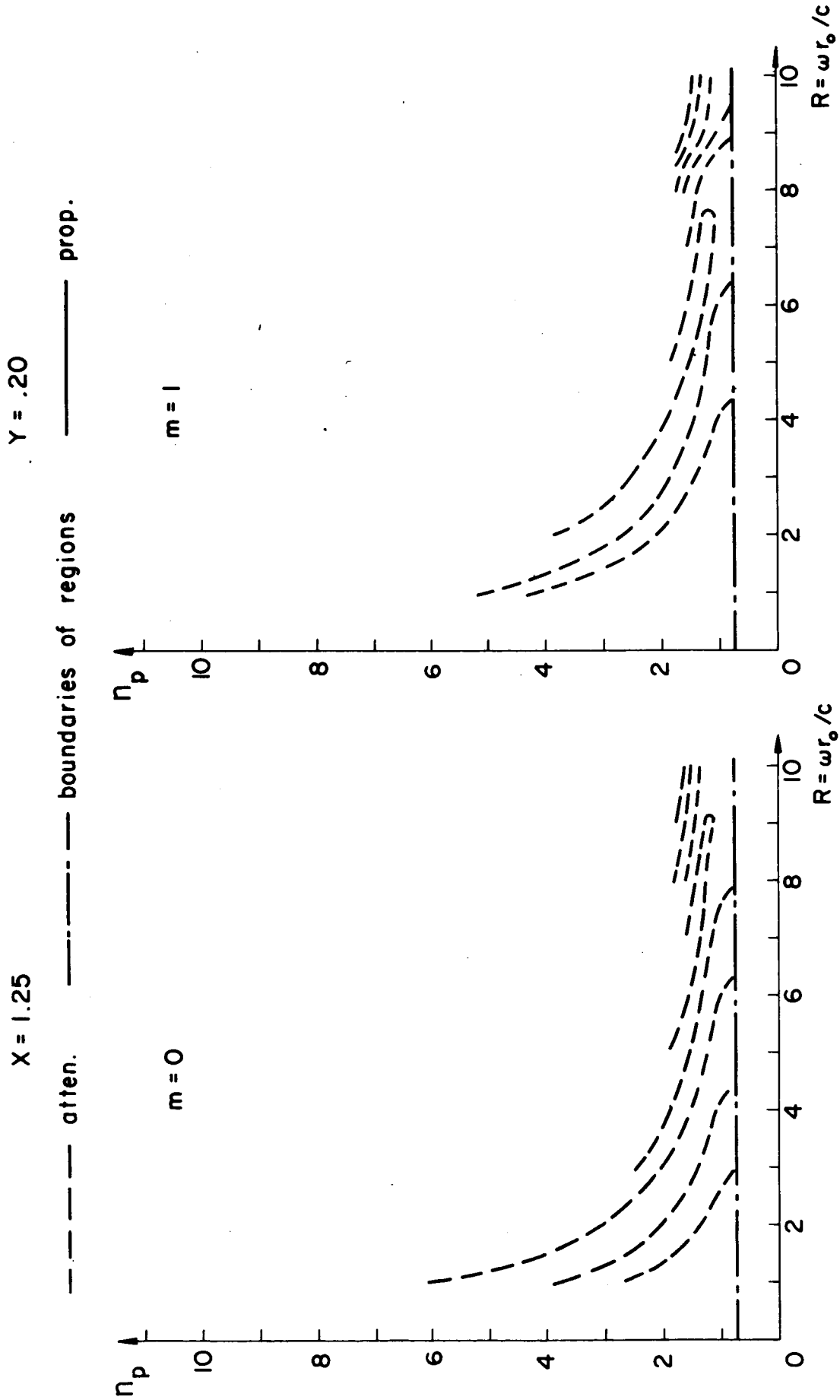


Figure (5-2) a) Continued

$Y = .80$
 ——— prop.

--- atten. --- boundaries of regions

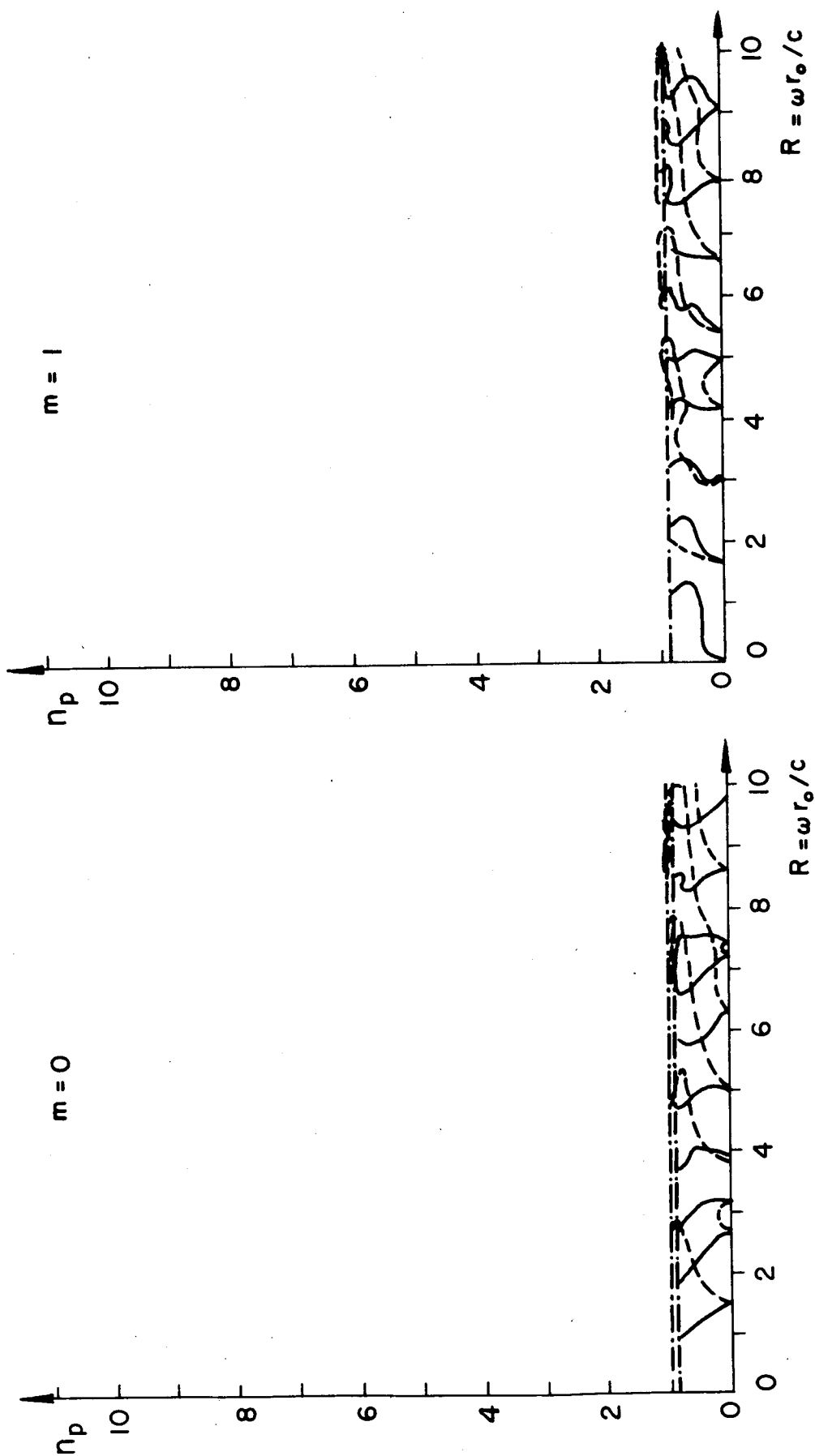


Figure (5-2) Continued

b) $X = 0.4; 0.8 \leq Y \leq 1.8$

$X = .40$

--- atten.

--- boundaries of regions

$Y = .90$

— prop.

n_p

n_p

$m = 0$

$m = 1$

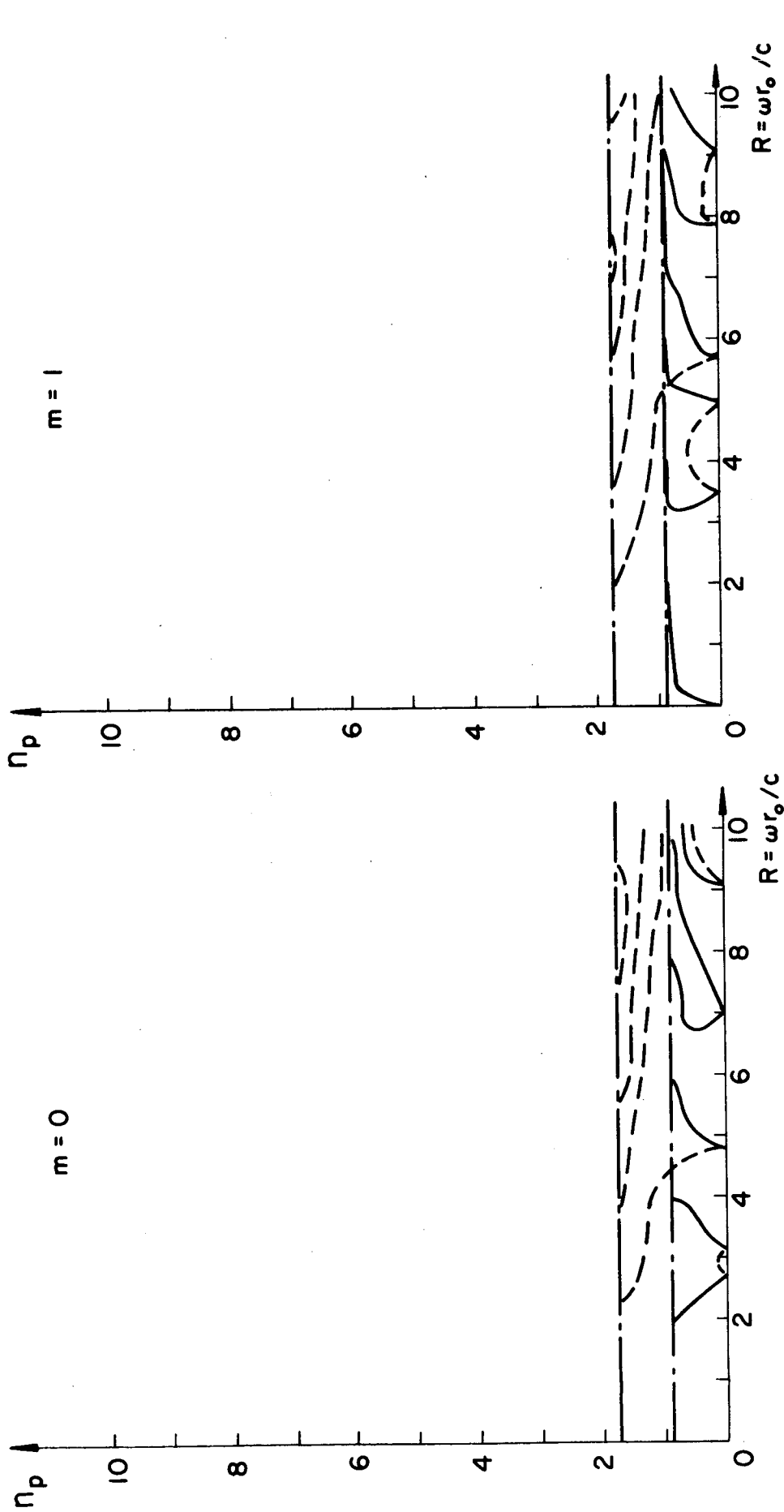


Figure (5-2) b) Continued

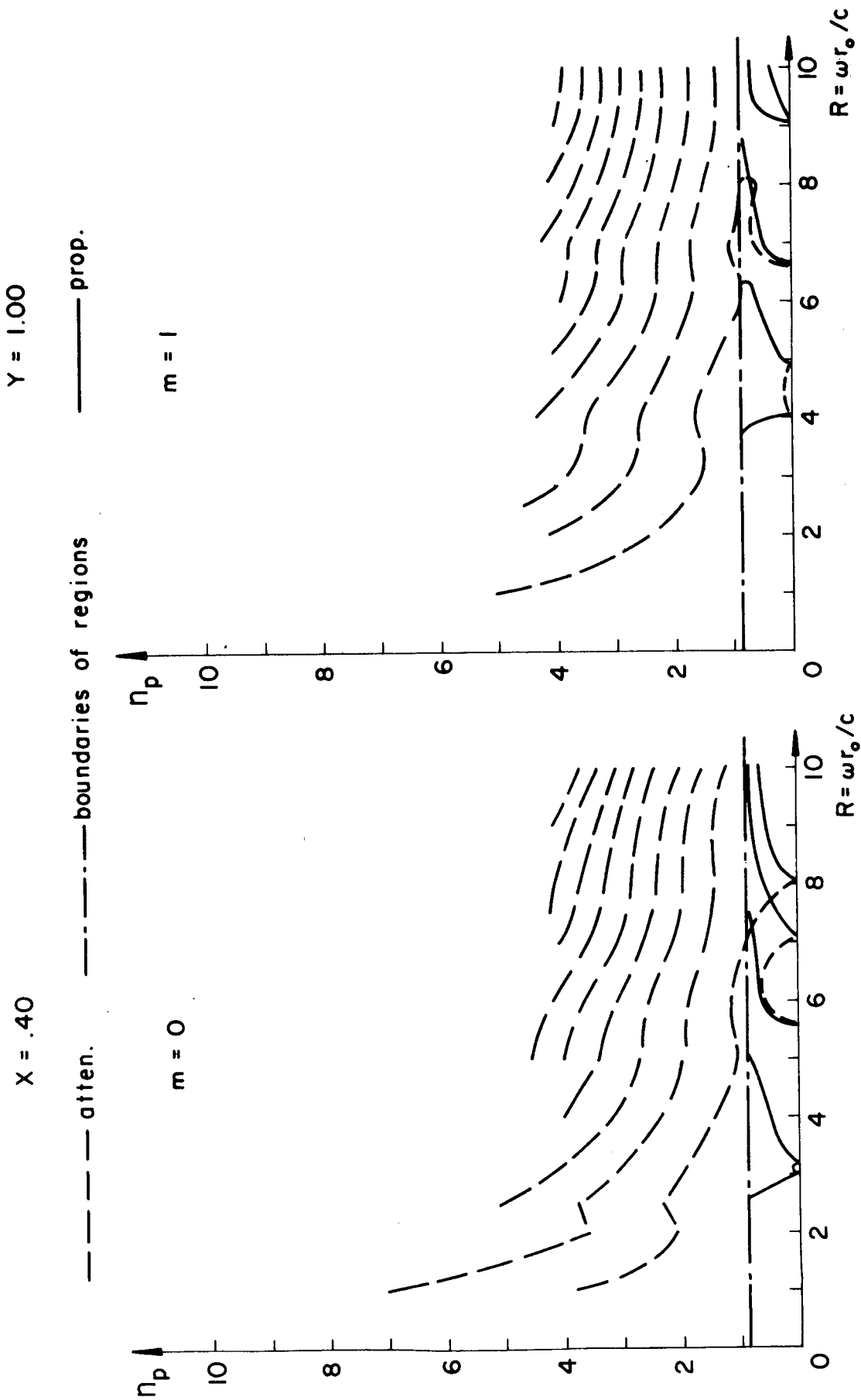


Figure (5-2) b) Continued

$X = .40$

$Y = 1.80$

--- atten. --- boundaries of regions --- prop.

n_p

n_p

$m = 0$

$m = 1$

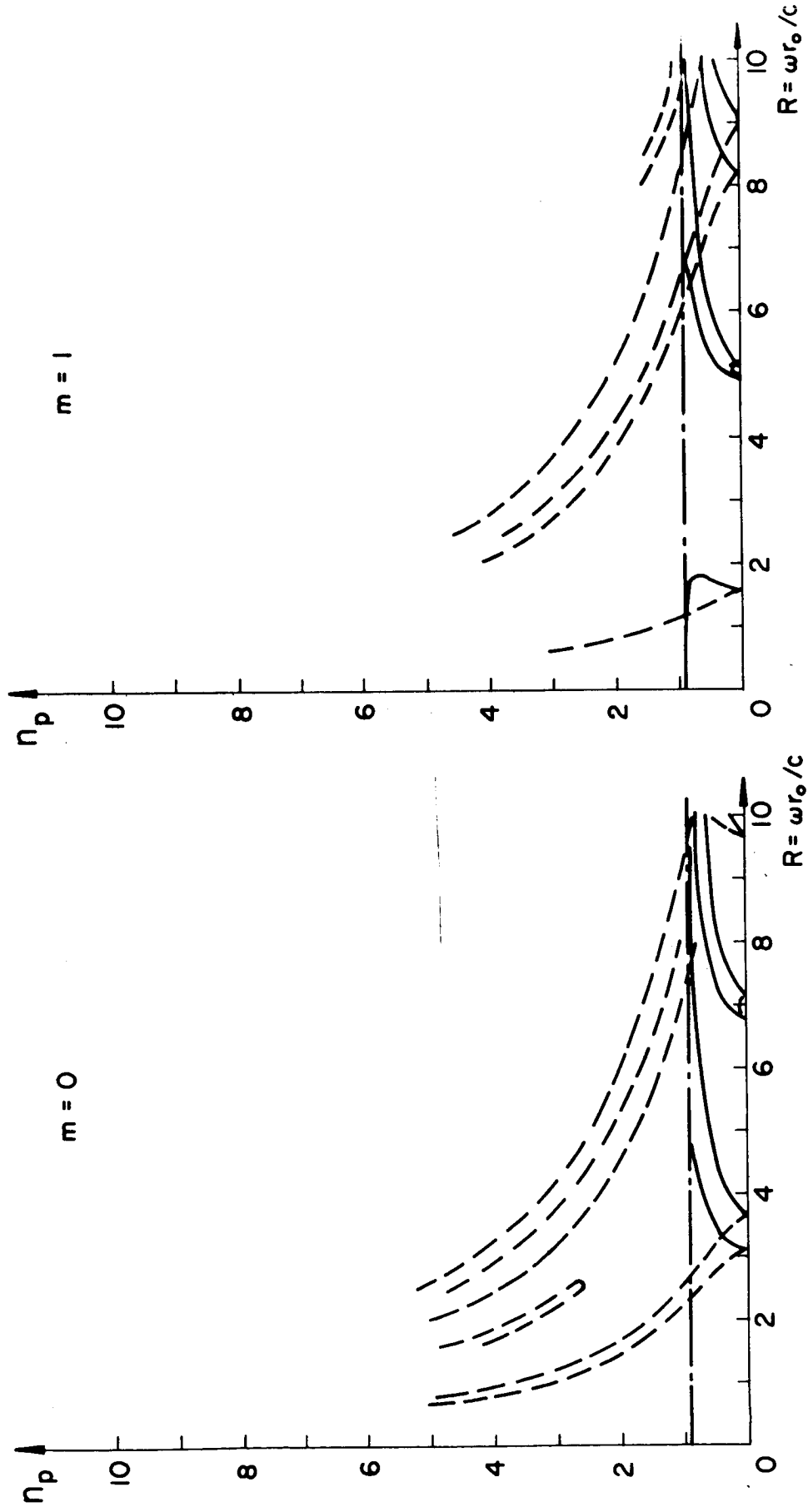


Figure (5-2) b) Continued

$X = 1.20$ $Y = 1.10$
 --- atten. --- boundaries of regions --- prop.

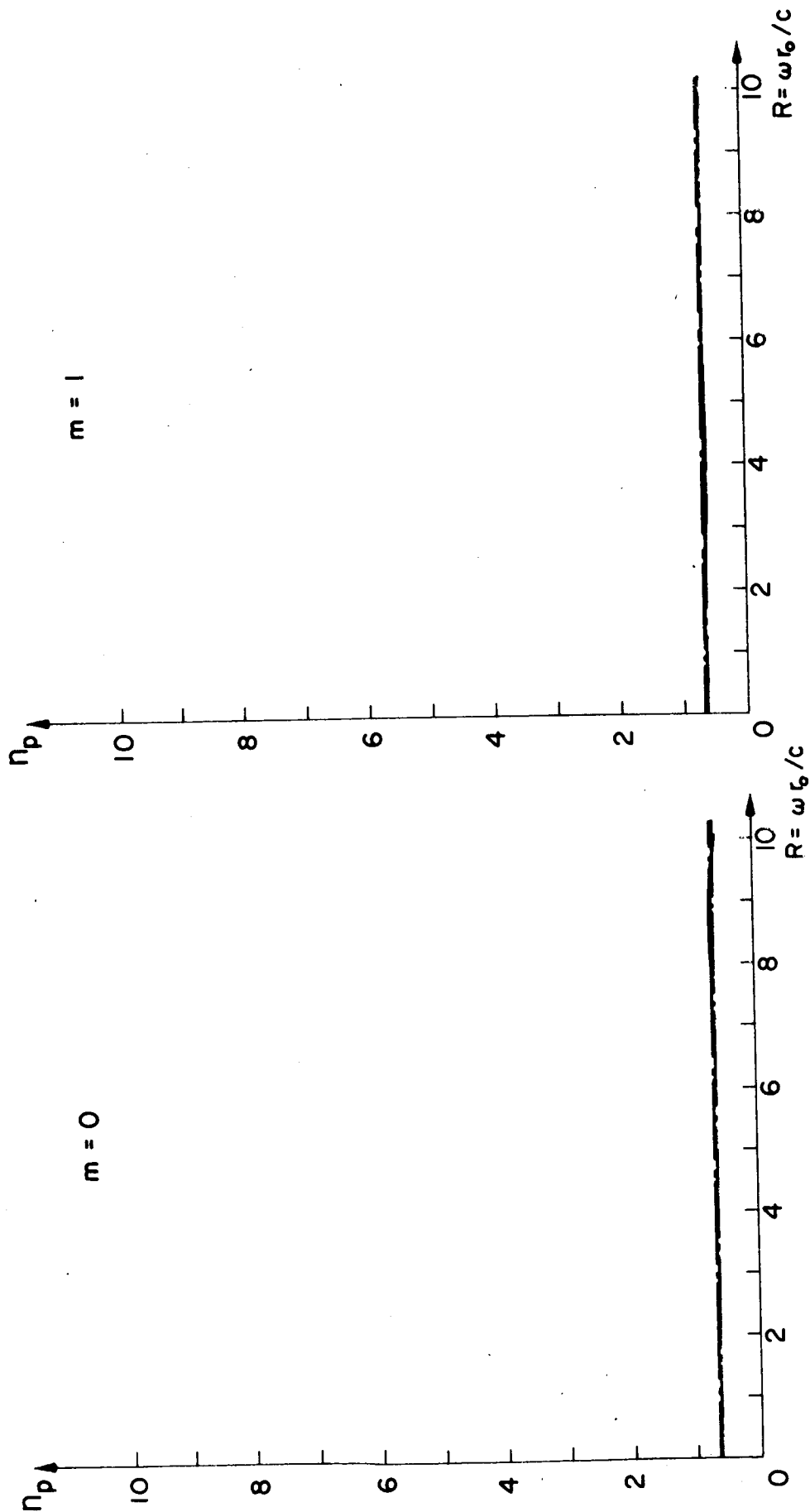


Figure (5-2) Continued
 c) $X = 1.2$; $Y = 1.1$; Attenuation occurs for $|n_p| > 10$ for $m = 0$ no propagation
 $m = 1$; $n_p = .7$ constantly

$$\omega_N^2 = \left(\frac{c}{l_0}\right)^2 20 \qquad \omega_H = \left(\frac{c}{l_0}\right) 2 \qquad M = 0 \qquad \text{---} = \text{PROPAGATION}$$

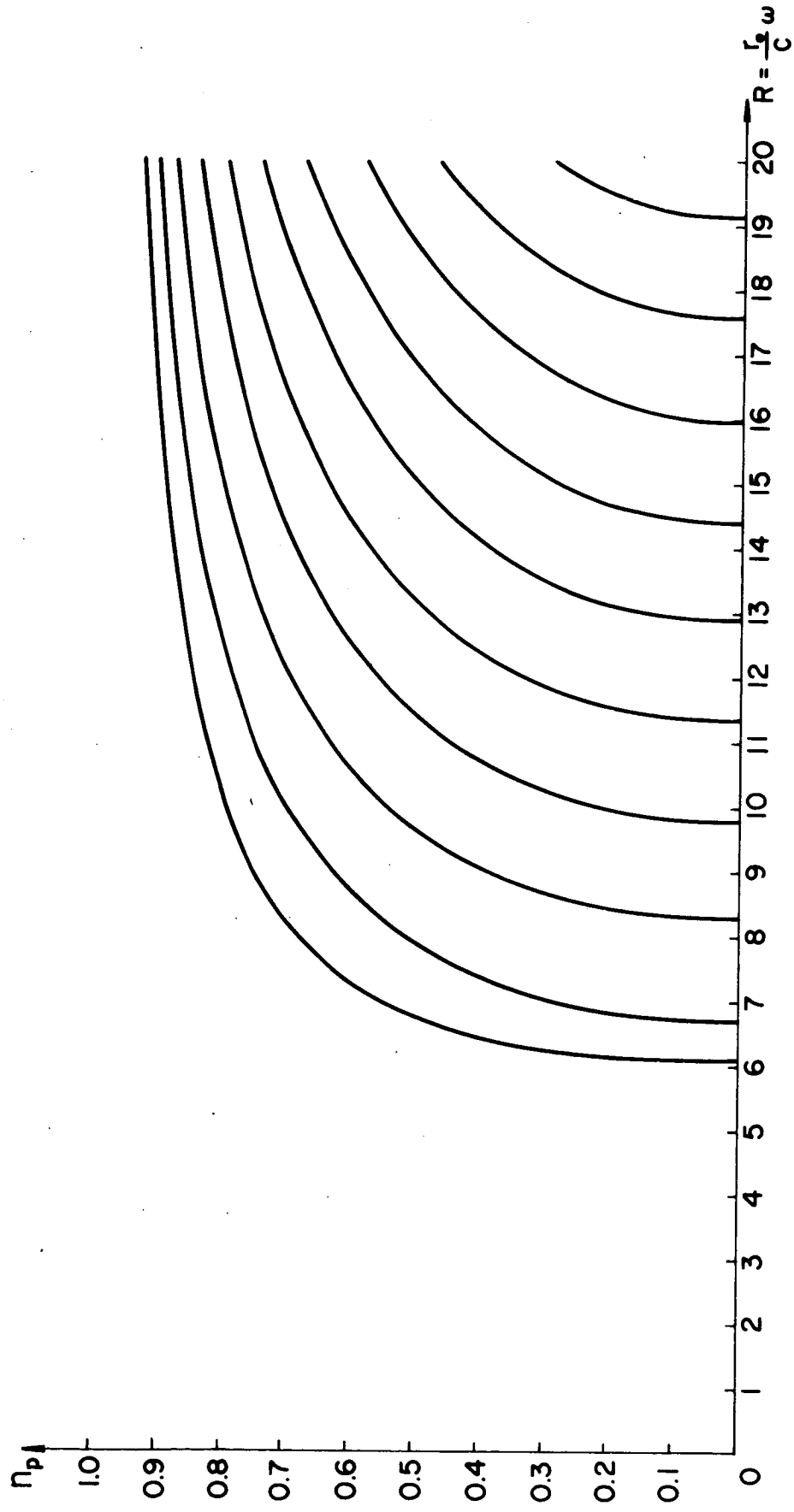


Figure (5-3) For a cold plasma filled guide n_p vs $R = r_0 \omega / c$ for fixed ω_H and ω_N .

a) $m = 0$

$$\omega_N^2 = \left(\frac{c}{r_0}\right)^2 20 \quad \omega_H = \left(\frac{c}{r_0}\right)^2 2 \quad M = 1 \quad \text{---} = \text{PROPAGATION}$$

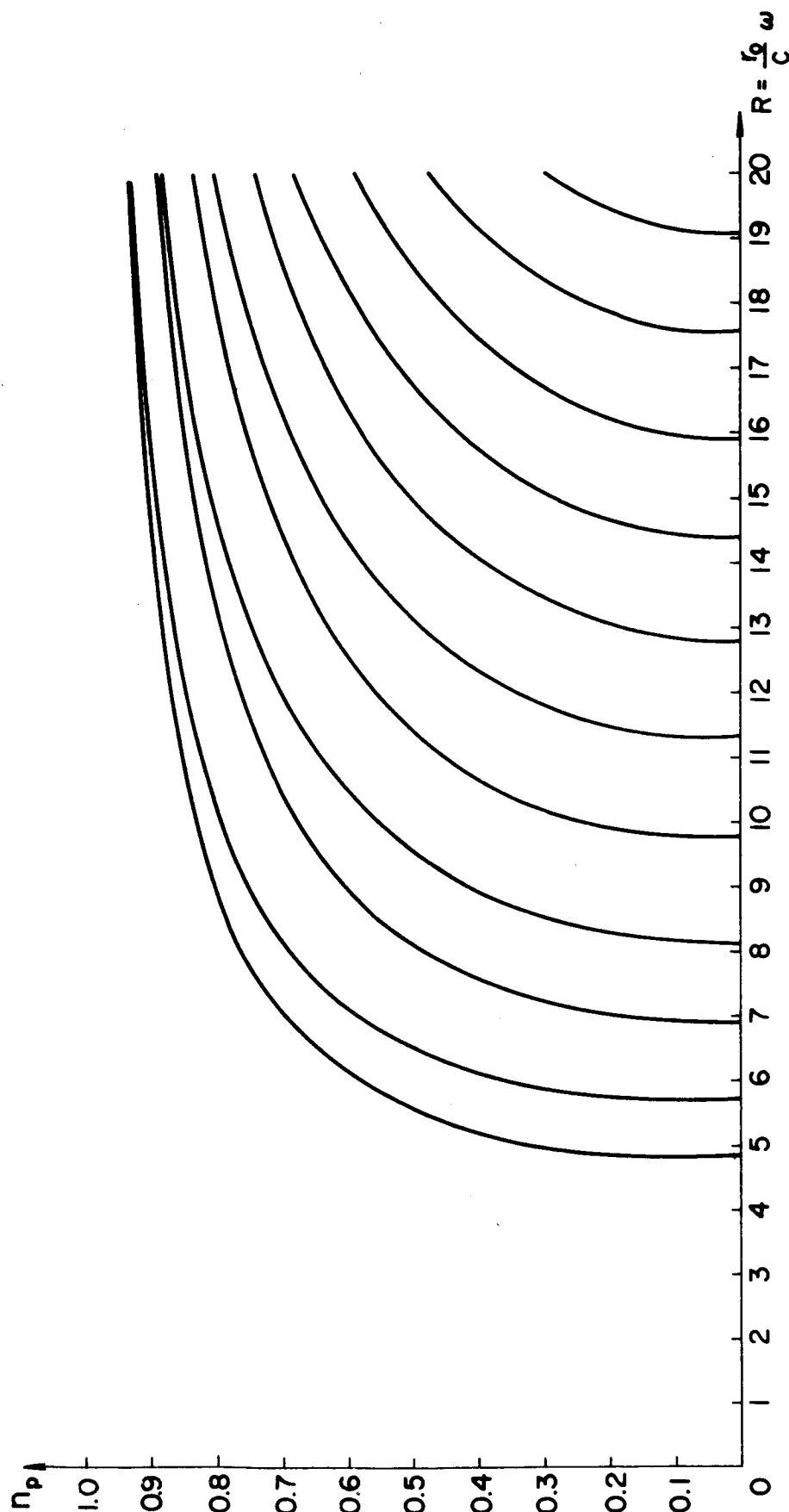


Figure (5-3) Continued b) $m = 1$

of W , X , and Y . Therefore, for this case, only those values of X and Y which are used in Figure (5-5) are considered. For fixed W , X and Y , one finds along the real axis of n_p^2 at most three regions in which real solutions for n_p^2 in the characteristic equation can take place. Figure (5-4) represents two sets of these regions as a function of X or Y where X/Y is kept constant for $W = 5 \times 10^{-6}$ and $W = 5 \times 10^{-4}$ respectively. For each set of fixed W , X and Y , the first of these regions has a negative upper bound with very large absolute value and a lower bound of $-\infty$. One can regard them as regions of very high attenuation. The second is also an attenuating region except that its upper and lower bounds are finite. The third can be called region of propagation, although its lower bound is generally negative with very small absolute value similar to the cold plasma case. This region also has a finite upper bound. It is again seen that for large X and Y there can be no real solution for n_p^2 in the characteristic equation. As X and Y decrease one begins to find regions of very high attenuation, then regions of moderate attenuation and regions of propagation. Figure (5-5) shows n_p as a function of $R = r_o \omega / c$ for fixed W , ω_n and ω_H . The computed solutions of the characteristic equation are marked with small circles and dots. As noted in Chapter III, the solutions can be classified into two categories. Those shown with circles in Figure (5-5a) through (5-5d) correspond to the so-called "quasi optical modes" and plasma modes. A comparison of these figures with Figure (5-3) reveals that most of these points lie very close to the curves in Figure (5-3). These points correspond to the quasi optical modes and their

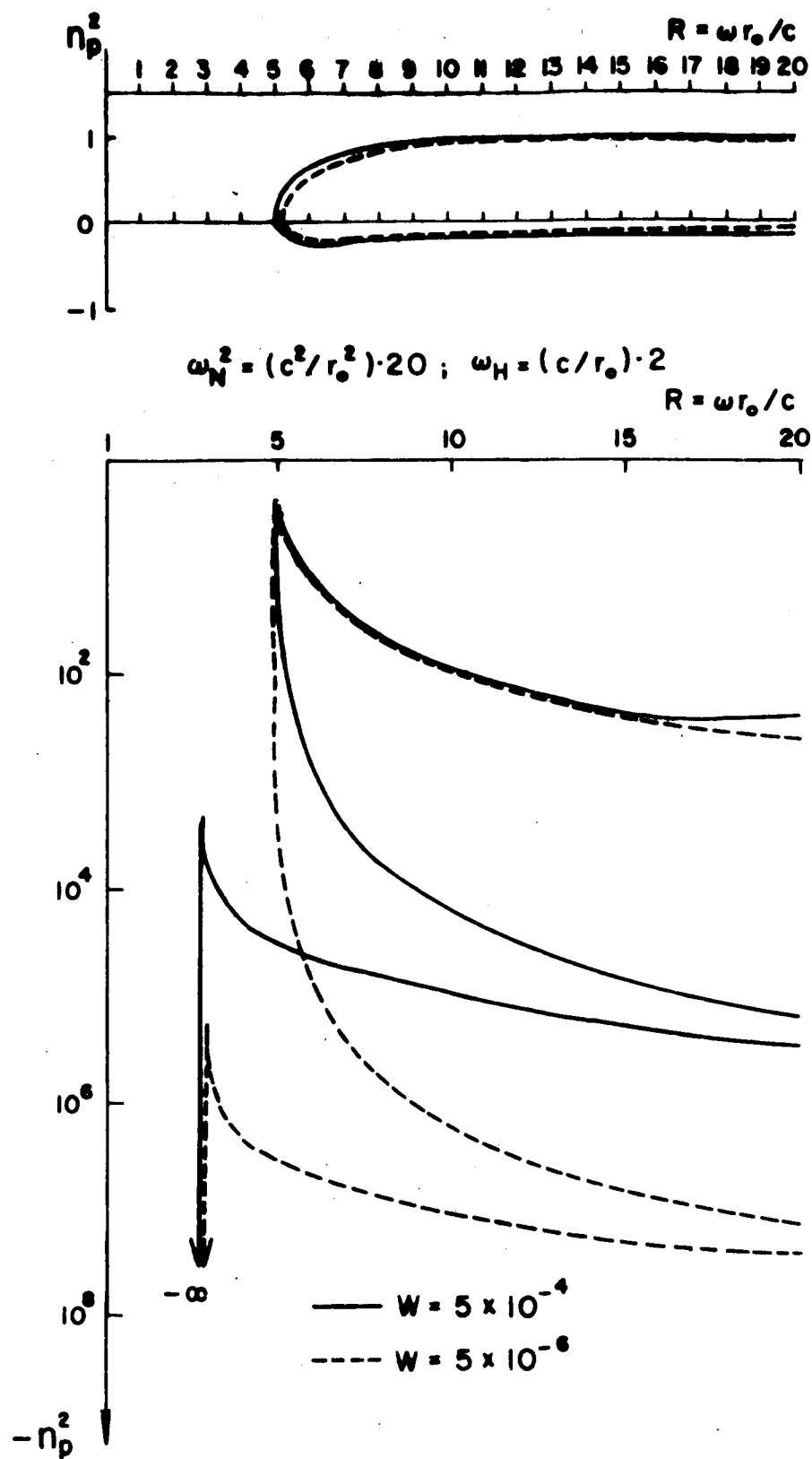


Figure (5-4) For a warm plasma filled guide, the regions of propagation with changing frequency

$$\omega_N^2 = \left(\frac{c}{r_0}\right)^2 20$$

$$\omega_H = \left(\frac{c}{r_0}\right)^2 2$$

$$W = 5 \times 10^{-6}$$

$$M = 0$$

n_p

• PROPAGATION

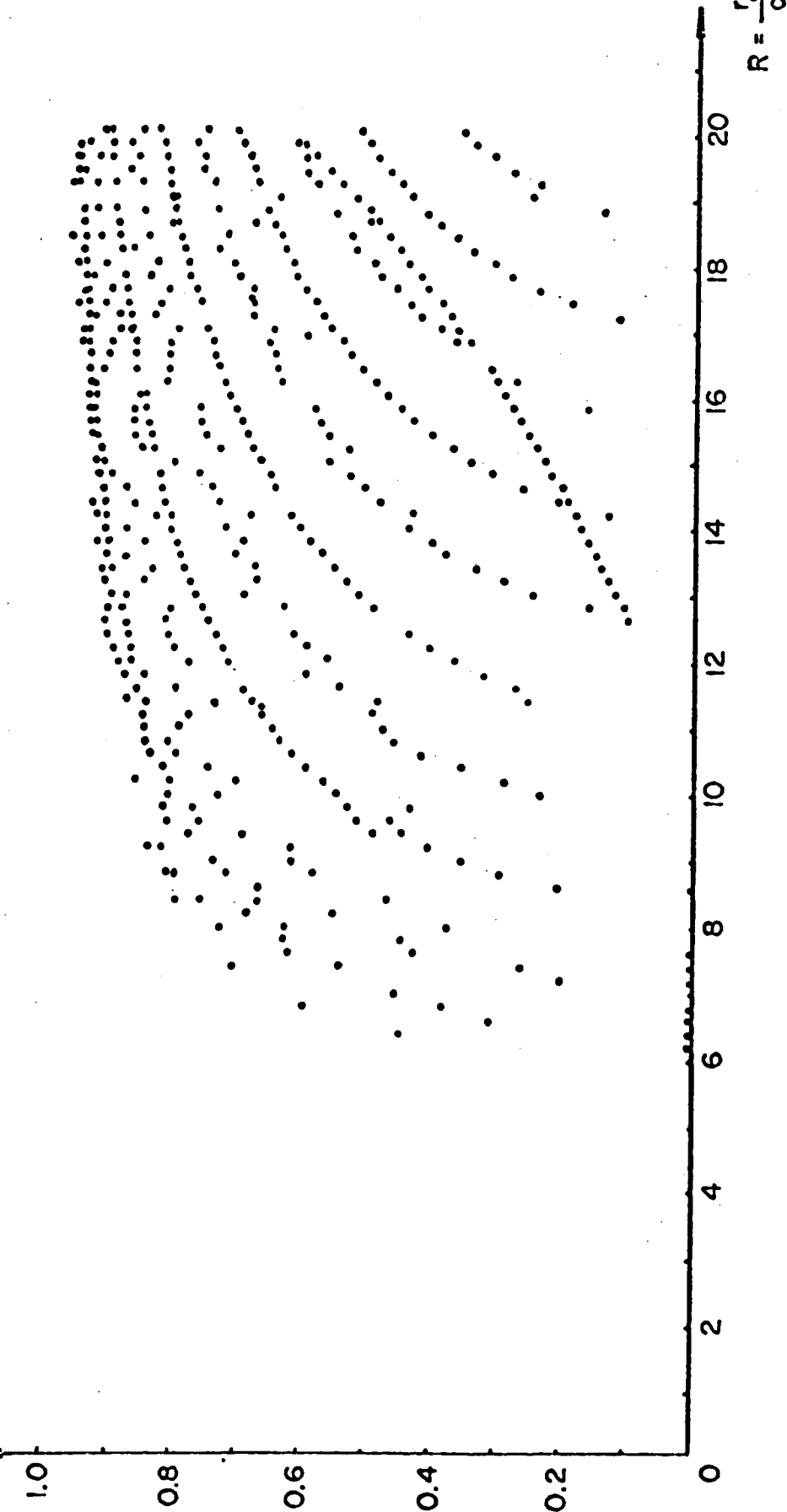


Figure (5-5) Computed real solutions of n_p vs. $R = r_0 \omega/c$ for a warm plasma-filled guide with fixed ω_N , ω_H and W .

a) $W = 5 \times 10^{-6}$; $m = 0$.

$$\omega_N^2 = \left(\frac{c}{r_0^2}\right)20 \qquad \omega_H = \left(\frac{c}{r_0}\right)2 \qquad W = 5 \times 10^{-6} \qquad M = 1$$

• PROPAGATION

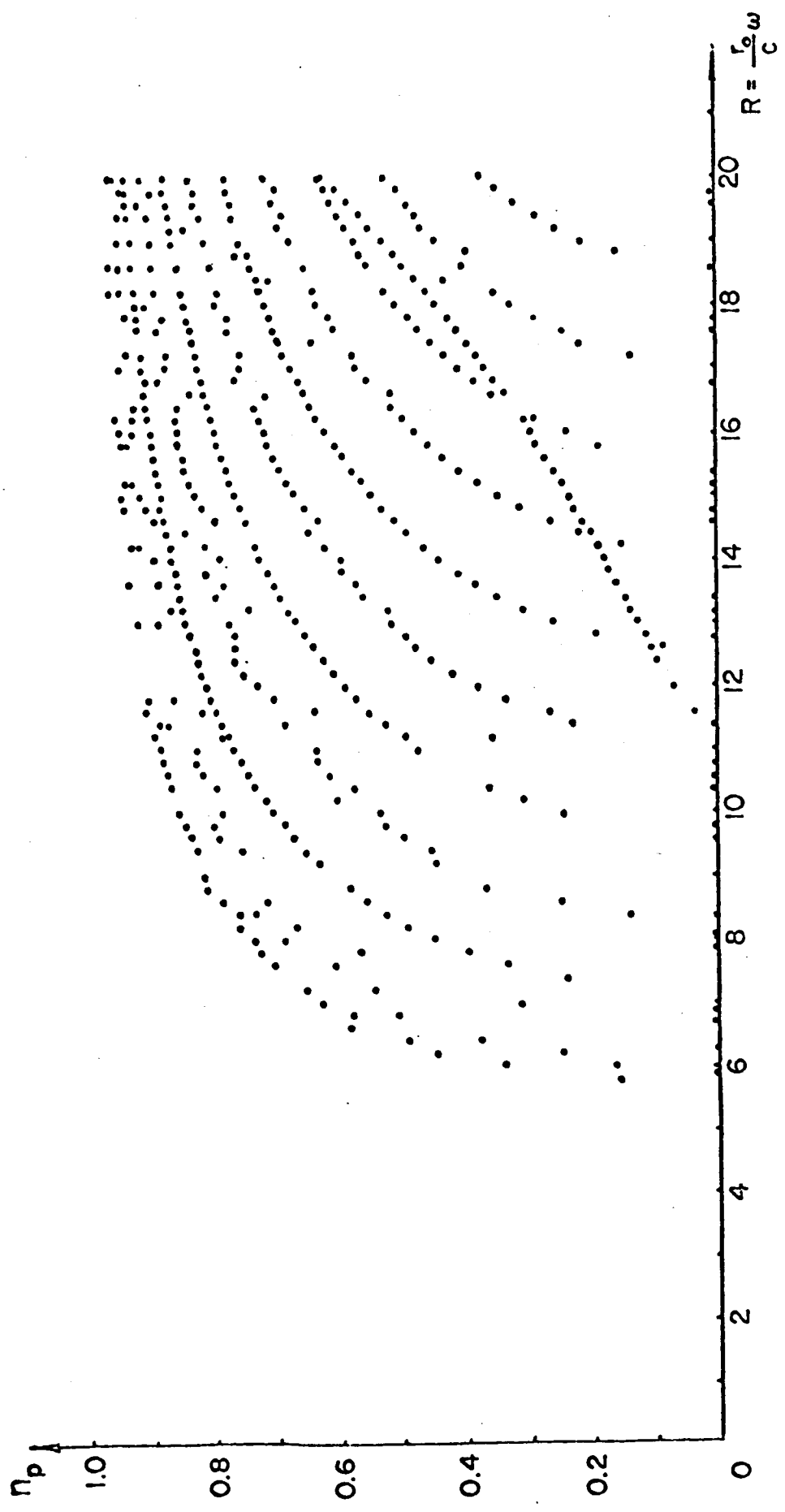


Figure (5-5) Continued

b) $W = 5 \times 10^{-6}$; $m = 1$

$$\omega_N^2 = \left(\frac{c}{r_0}\right)^2 20$$

$$\omega_H = \left(\frac{c}{r_0}\right)^2 2$$

$$W = 5 \times 10^{-4}$$

$$M = 0$$

• PROPAGATION

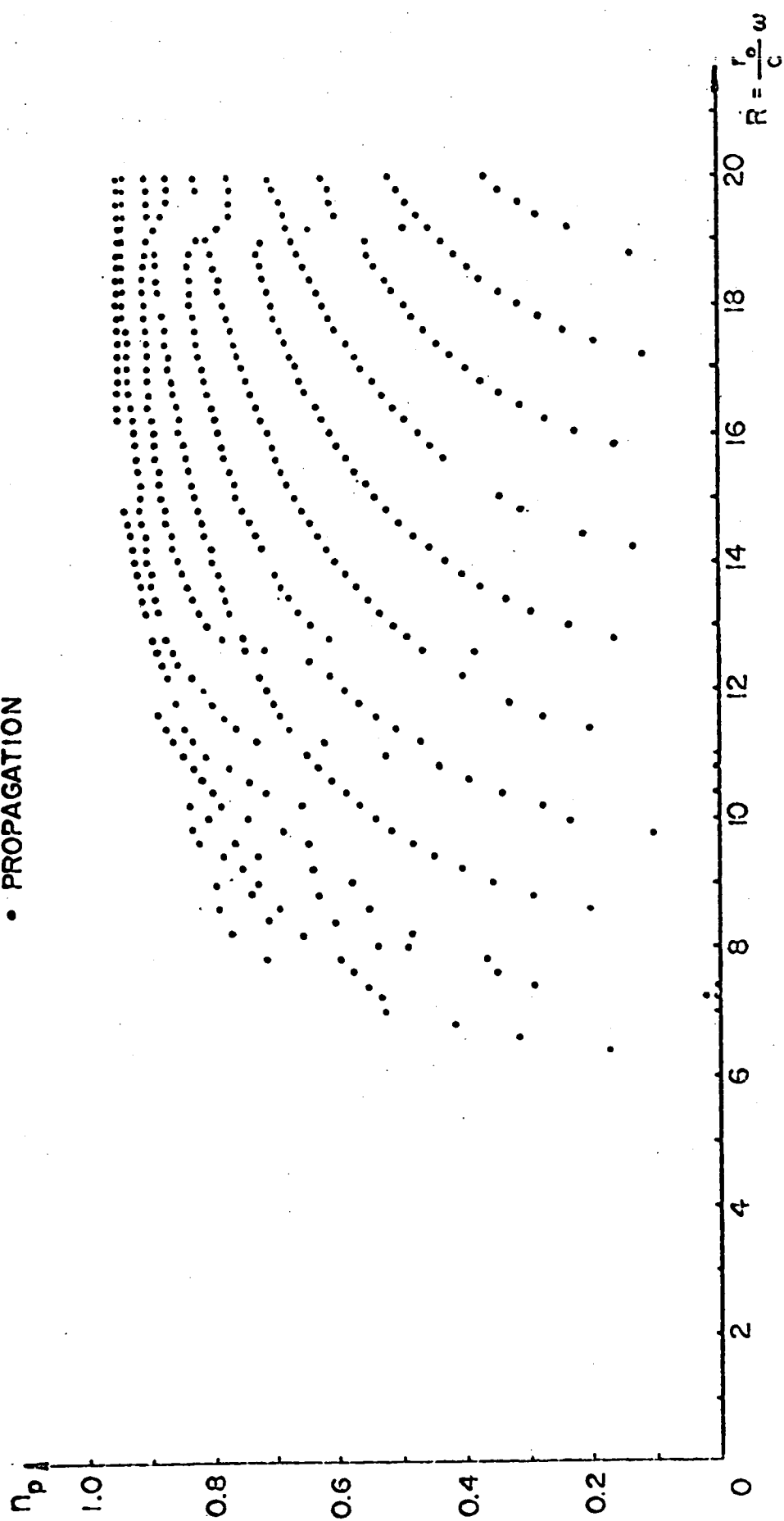


Figure (5-5) Cont Inued

c) $W = 5 \times 10^{-4}$; $m = 0$

$$\omega_N^2 = \left(\frac{c}{r_0}\right)^2 20 \quad \omega_H = \left(\frac{c}{r_0}\right)^2 2 \quad W = 5 \times 10^{-4} \quad M = 1$$

• PROPAGATION

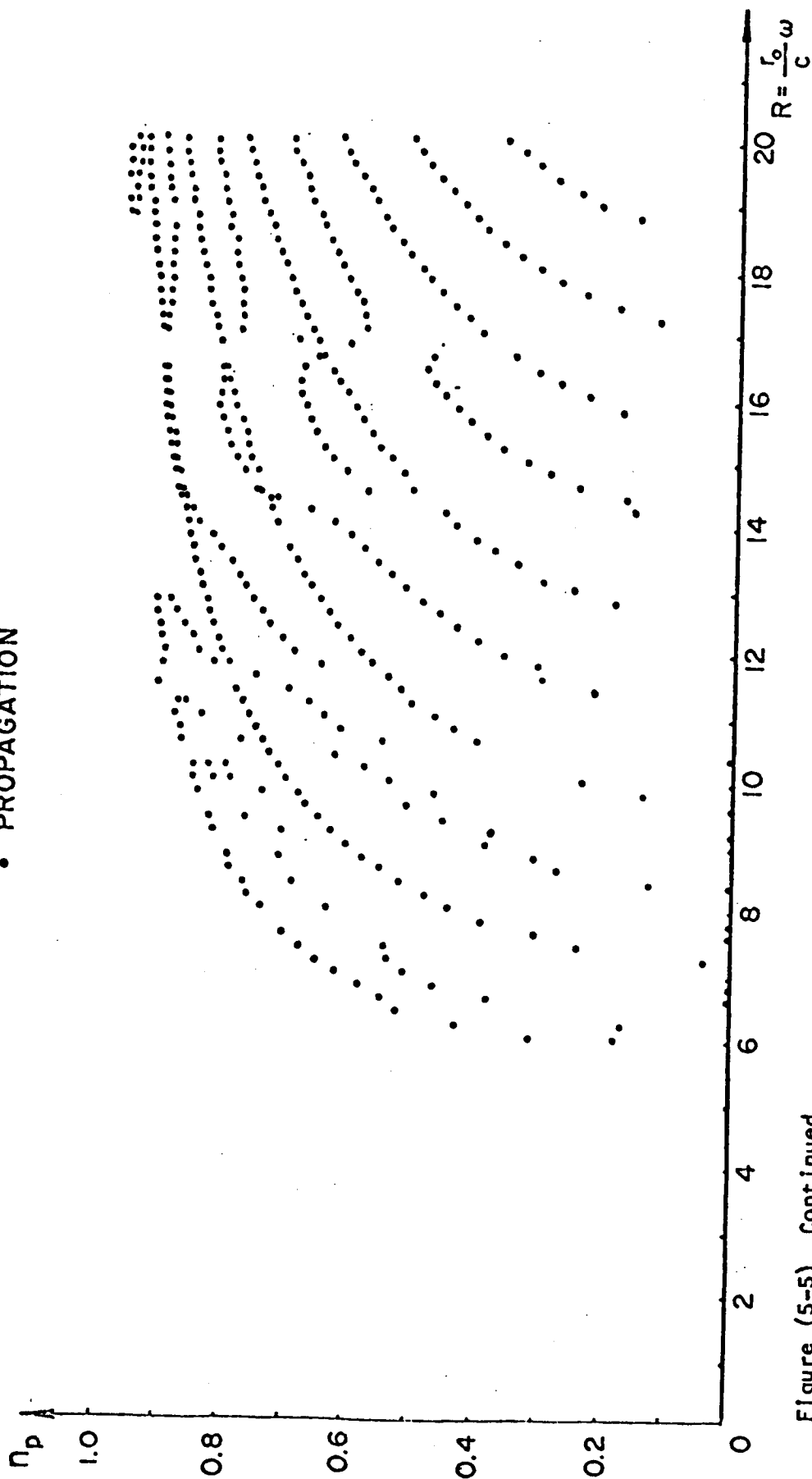


Figure (5-5) Continued

d) $W = 5 \times 10^{-4}$; $m = 1$

$$\omega_N^2 = \left(\frac{c}{r_0^2}\right) 20$$

$$\omega_H = \left(\frac{c}{r_0}\right) 2$$

$$W = 5 \times 10^{-4}$$

$$M = 0$$

— PROPAGATION

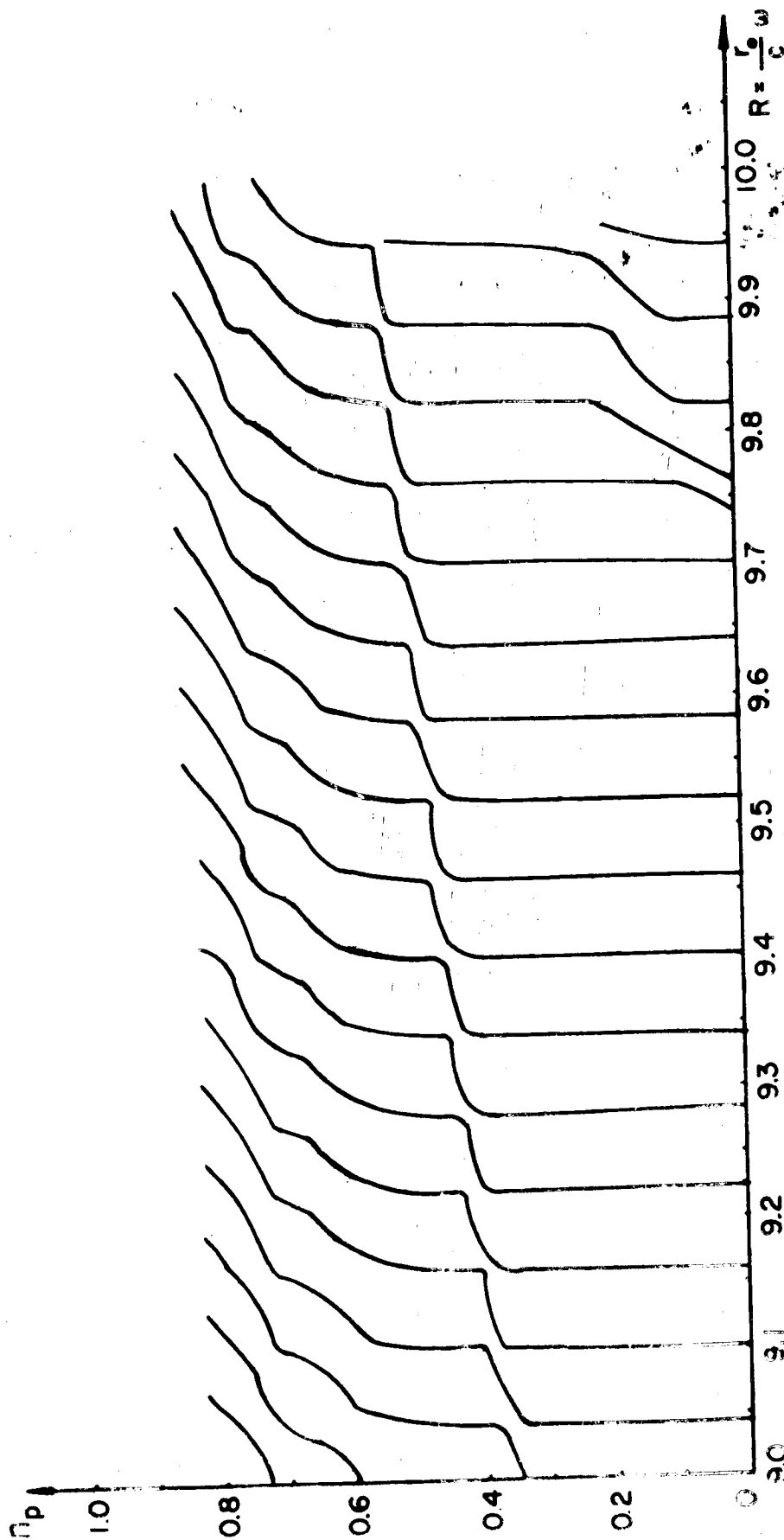


Figure (5-5) Continued

a) Part of the η_p vs $R = r_0 \omega / c$ diagram for $W = 5 \times 10^{-4}$; $m = 0$ at $q \leq R \leq 10$.

values are only slightly perturbed by the introduction of compressibility to the plasma, they are roughly independent of the electron temperature. The points which correspond to the so-called plasma modes, as already indicated in Eqs. (3-38) and (3-33.5) lie on a very dense family of nearly vertical lines. These points, being too densely packed, are not shown on these figures (except for a few as shown by dots and which correspond to transitions from plasma to quasi-optical modes*). Instead, as an example, a small section of Figure (5-5c) is magnified many times as indicated in Figure (5-5e) where the propagation constants of the plasma modes lie on a family of nearly vertical lines. The sharp bends at the ends of these lines are resulted from the strong coupling between the plasma modes and quasi-optical modes.

*In searching for the zeros of the characteristic equation, as those shown in Figures (5-5a) through (5-5d), we have divided the abscissa, namely R , into equally spaced grids. Since the loci of n_p of the hybrid modes form a set of nearly vertical lines, their intersections with the constant R grids will give us some of the solutions for n_p . Some of these are shown in the aforementioned figures. Clearly these points, being the solutions on various loci of n_p , should not be connected by a curve. If the grids along R changed, the intersections will be changed accordingly. In other words, these points are but a small part of the solutions for n_p .

VI. MODAL WAVES DUE TO A CURRENT SOURCE IN THE WAVEGUIDE

In this chapter we consider the field and power carried in various modes due to a given electric current and particle source. To this end we first apply Eqs. (4-1) and (4-2) to a section of the guide which encloses the sources as shown in Figure (6-1). The surface S of this section is composed of two transverse planes S_1 and S_2 at $z = z_1$, and z_2 and the guide wall. For simplicity we again assume that the static magnetic field is parallel to z^1 , namely the guide axis. After invoking the boundary conditions one obtains:

$$\begin{aligned}
 & - \iint_{S_1} \{ [\tilde{E}_q \times \tilde{H}_n^* - T_e(N_q/N_o) \tilde{I}_n^*] + [\tilde{E}_n^* \times \tilde{H}_q - T_e(N_n^*/N_o) \tilde{I}_q] \} \cdot d\tilde{S} \\
 & + \iint_{S_2} \{ [\tilde{E}_q \times \tilde{H}_n^* - T_e(N_q/N_o) \tilde{I}_n^*] \\
 & + [\tilde{E}_n^* \times \tilde{H}_q - T_e(N_n^*/N_o) \tilde{I}_q] \} \cdot d\tilde{S} = U(q, n)
 \end{aligned} \quad (6-1.1)$$

where

$$\begin{aligned}
 U(q, n_{pn}) = & \iiint_V [- \tilde{J}_q \cdot \tilde{e}_n^* \exp(\gamma_n^* z) - \tilde{K}_q \cdot \tilde{h}_n^* \exp(\gamma_n^* z) \\
 & - e T_e n^* \exp(\gamma_n^* z) j \omega \rho_q + (1/e) \tilde{i}_n^* \exp(\gamma_n^* z) \cdot \tilde{F}_q] dV.
 \end{aligned} \quad (6-1.2)$$

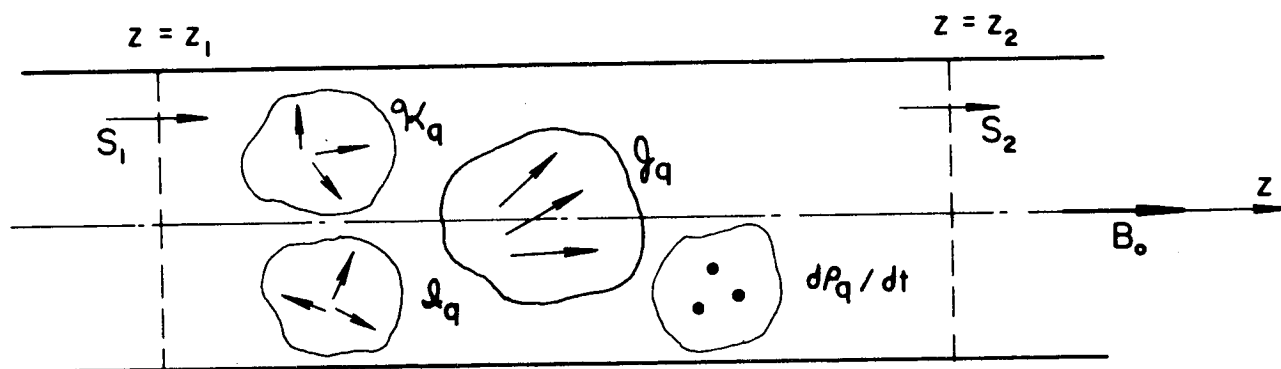


Figure (6-1) Longitudinal cross section of a waveguide with sources

With all the sources "q" one may associate a potential function

$$\pi_{qk} = \sum_{m, \ell} \lambda_{m\ell k} \sum_j \delta_j \pi_j (n_{pm\ell}) \exp(-jk_o n_{pm\ell k} z + jm\phi) \quad (6-2)$$

where ℓ and m denote the mode indices, $j = 1, 2$ for cold plasma and $j = 1, 2, 3$ for warm plasma. In general, the fields on the two sides of the current sources "q", in the guide, are different from each other. Therefore, to the coefficients $\lambda_{m\ell}$ a subscript k is added so that $k = 1$ for $z \leq z_1$ and $k = 2$ for $z \geq z_2$.

Let us consider the matrix (M_j) which is defined in Eqs. (3-5.2) and (3-23.5) for cold and warm plasmas respectively. In these matrices the first two elements of the third row and the first two elements of the third column are zero. Let us define submatrices (M_{jt}) which are made of the first two rows and the first two columns of the matrices (M_j) . Let the transverse field components on S_1 be E_{q1t} , E_{n1t} and H_{q1t} , H_{n1t} and the longitudinal convection current components be I_{q1z} and I_{n1z} and the perturbed electron densities be N_{q1} , N_{n1} . Similarly let all these quantities on S_2 be denoted in the same manner except that the subscript 1 is replaced by 2. Now let the operator ∇_t be defined as

$$\nabla_t = \nabla - d_z \frac{\Lambda}{z} = \frac{\Lambda}{r} d_r + \frac{\Lambda}{\phi} (1/r) d_\phi$$

Then, for the quantities with subscript q , one can write the following expressions:

$$\begin{aligned} \tilde{E}_{qkt} = -jk_o \sum_{m,l} \lambda_{mlk} n_{pm\ell k} \sum_j [M_{jt} \cdot \delta_j \nabla_t \pi_j (n_{pm\ell k})] \\ \begin{matrix} z = 0 \\ \varphi = 0 \end{matrix} \\ \exp(-jk_o n_{pm\ell k} z_k + jm\varphi) \end{aligned} \quad (6-3.1)$$

$$\begin{aligned} \tilde{H}_{qkt} = [jk_o / (\omega \mu_o)] \sum_{m,l} \lambda_{mlk} \sum_j [K_{jt} \cdot \delta_j \nabla_t \pi_j (n_{pm\ell k}) \exp(-jk_o n_{pm\ell k} z_k + jm\varphi)] \\ \begin{matrix} z = 0 \\ \varphi = 0 \end{matrix} \end{aligned} \quad (6-3.2)$$

$$\begin{aligned} \tilde{I}_{qkz} = -jk_o \omega \epsilon_o \sum_{m,l} \lambda_{mlk} n_{pm\ell k} \sum_j V_{33j} \delta_j \pi_j (n_{pm\ell k}) \exp(-jk_o n_{pm\ell k} z_k + jm\varphi) \\ \begin{matrix} z = 0 \\ \varphi = 0 \end{matrix} \end{aligned} \quad (6-3.3)$$

$$\begin{aligned} N_{qk} = -j(\epsilon_o k_o^2 / e) \sum_{m,l} \lambda_{mlk} \sum_j R_j \delta_j \pi_j (n_{pm\ell k}) \exp(-jk_o n_{pm\ell k} z_k + jm\varphi) \\ \begin{matrix} z = 0 \\ \varphi = 0 \end{matrix} \end{aligned} \quad (6-3.4)$$

where

$$R_j = n_{tj} (n_{tj} C_{1j} - n_{pm\ell k} C_{3j}). \quad (6-3.5)$$

In the above expressions if n_{ml} is real one has

$$\lambda \omega / \lambda n_{pm\ell k} \quad \begin{cases} < 0 & \text{for } k = 1 \\ > 0 & \text{for } k = 2, \end{cases} \quad (6-3.6.1)$$

$$(6-3.6.2)$$

whereas for purely imaginary $n_{pm\ell}$

$$jn_{pm\ell k} \begin{cases} < 0 & \text{for } k = 1 \\ > 0 & \text{for } k = 2. \end{cases} \quad (6-3.7.1)$$

$$(6-3.7.2)$$

Let the field with subscript n be one of the modes obtained earlier having the following quantities:

$$n_{pn} = n_{pvu}$$

$$\tilde{E}_{nt} = -jk_o n_{pvu} \sum_j M_{jt} \cdot \delta_j \nabla_t \pi_j (n_{pvu}) \exp(-jk_o n_{pvu} z + jv\varphi) \quad (6-3.8)$$

$$\begin{matrix} z = 0 \\ \varphi = 0 \end{matrix}$$

$$\tilde{H}_{nt} = j[k_o / (\omega \mu_o)] \sum_j K_{jt} \cdot \delta_j \nabla_t \pi_j (n_{pvu}) \exp(-jk_o n_{pvu} z + jv\varphi) \quad (6-3.9)$$

$$\begin{matrix} z = 0 \\ \varphi = 0 \end{matrix}$$

$$I_{nz} = -jk_o \omega \epsilon_o n_{pvu} \sum_j V_{33j} \delta_j \pi_j (n_{pvu}) \exp(-jk_o n_{pvu} z + jv\varphi) \quad (6-3.10)$$

$$\begin{matrix} z = 0 \\ \varphi = 0 \end{matrix}$$

$$N_n = -j(\epsilon_o k_o^2 / e) \sum_j R_j \delta_j \pi_j (n_{pvu}) \exp(-jk_o n_{pvu} z + jv\varphi) \quad (6-3.11)$$

$$\begin{matrix} z = 0 \\ \varphi = 0 \end{matrix}$$

Inserting the expressions given by Eqs (6-3) into Eq. (6-1)

one finds:

$$-\iint_{S_1} [k_o^2 / (\omega \mu_o)] \left[\sum_{m,l} \lambda_{ml} n_{pml} \sum_j M_{jt} \cdot \delta_j \nabla_t \pi_j (n_{pml}) \exp(-jk_o n_{pml} z_1 + jm\varphi) \right]$$

$z = 0$
 $\varphi = 0$

$$\times \left[\sum_j K_{jt}^* \cdot \delta_j^* \nabla_t \pi_j^* (n_{pvu}) \exp(jk_o n_{pvu}^* z_1 - jv\varphi) \right]$$

$z = 0$
 $\varphi = 0$

$$-[T_e k_o^2 \epsilon_o^2 \omega / (eN_o)] \left[\sum_{m,l} \lambda_{ml} \sum_j R_j \delta_j \pi_j (n_{pml}) \exp(-jk_o n_{pml} z_1 + jm\varphi) \right]$$

$z = 0$
 $\varphi = 0$

$$\cdot [n_{pvu}^* \sum_j V_{3j}^* \delta_j^* \pi_j^* (n_{pvu}) \exp(jk_o n_{pvu}^* z_1 - jv\varphi)]$$

$z = 0$
 $\varphi = 0$

$$-[k_o^2 / (\omega \mu_o)] n_{pvu}^* \left[\sum_j M_{jt}^* \delta_j^* \nabla_t \pi_j^* (n_{pvu}) \exp(jk_o n_{pvu}^* z_1 - jv\varphi) \right]$$

$z = 0$
 $\varphi = 0$

$$\times \left[\sum_{m,l} \lambda_{ml} \sum_j K_{jt} \cdot \delta_j \nabla_t \pi_j (n_{pml}) \exp(-jk_o n_{pml} z_1 + jm\varphi) \right]$$

$z = 0$
 $\varphi = 0$

$$-[T_e k_o^2 \epsilon_o^2 \omega / (eN_o)] \left[\sum_j R_j^* \delta_j^* \pi_j^* (n_{pvu}) \exp(jk_o n_{pvu}^* z_1 - jv\varphi) \right]$$

$z = 0$
 $\varphi = 0$

$$- [T_e k_o \epsilon_o^2 \omega / (e N_o)] \lambda_{vu2} \left[\sum_j R_j \delta_j \pi_j (n_{pvu}) \right]_{\substack{z=0 \\ \varphi=0}} n_{pvu} \left[\sum_j V_{33j}^* \delta_j^* \pi_j^* (n_{pvu}) \right]_{\substack{z=0 \\ \varphi=0}}$$

$$- [k_o / (\omega \mu_o)] n_{pvu} \left[\sum_j M_{jt}^* \delta_j^* \nabla_t \pi_j^* (n_{pvu}) \right]_{\substack{z=0 \\ \varphi=0}} \times \lambda_{vu2} \left[\sum_j K_{jt} \delta_j \nabla_t \pi_j (n_{pvu}) \right]_{\substack{z=0 \\ \varphi=0}}$$

$$- [T_e k_o \epsilon_o^2 \omega / (e N_o)] \left[\sum_j R_j^* \delta_j^* \pi_j^* (n_{pvu}) \right]_{\substack{z=0 \\ \varphi=0}} \lambda_{vu2} n_{pvu} \left[\sum_j V_{33j} \delta_j \pi_j (n_{pvu}) \right]_{\substack{z=0 \\ \varphi=0}} \cdot dS_{\sim}$$

$$= U(q, n_{pvu}) / (n_{pvu} k_o)$$

$$\text{or } - [2k_o / (\omega \mu_o)] \lambda_{vu2} \iint_s \text{Re} \left\{ \left[\sum_j M_{jt} \delta_j \nabla_t \pi_j (n_{pvu}) \right]_{\substack{z=0 \\ \varphi=0}} \right\} \times$$

$$\left[\sum_j K_{jt}^* \delta_j^* \nabla_t \pi_j^* (n_{pvu}) \right]_{\substack{z=0 \\ \varphi=0}} \cdot dS_{\sim}$$

$$- [2T_e k_o \epsilon_o^2 \omega / (e N_o)] \lambda_{vu2} \iint_s \text{Re} \left\{ \left[\sum_j R_j \delta_j \pi_j (n_{pvu}) \right]_{\substack{z=0 \\ \varphi=0}} \right\} \times$$

$$\left[\sum_j V_{33j}^* \delta_j^* \pi_j^* (n_{pvu}) \right]_{\substack{z=0 \\ \varphi=0}} \cdot dS_{\sim}$$

$$= U(q, n_{pvu}) / (n_{pvu} k_o)$$

$$n_{pvu} = \text{Real}$$

$$\partial\omega/\partial n_{pvu} > 0 \quad (6-5.1)$$

(a2)

$$\partial\omega/\partial n_{pvu} < 0$$

which indicates a mode carrying energy in the -z direction. For this case, Eq. (6-4) assumes the form

$$[2k_o/(\omega\mu_o)]\lambda_{vu1} \iint_S \text{Re}\left\{\left[\sum_j M_{jt} \cdot \delta_j \nabla_t \pi_j(n_{pvu})\right] \times \left[\sum_j K_{jt}^* \delta_j^* \nabla_t \pi_j(n_{pvu})\right]\right\} dS$$

$z = 0$
 $\varphi = 0$

$$+ [2T_e k_o \epsilon_o^2 \omega / (eN_o)] \lambda_{vu1} \iint_S \text{Re}\left\{\left[\sum_j R_j \delta_j \pi_j(n_{pvu})\right] \cdot \left[\sum_j V_{33j}^* \delta_j^* \pi_j^*(n_{pvu})\right]\right\} dS$$

$z = 0$
 $\varphi = 0$ $z = 0$
 $\varphi = 0$

$$= U(q, n_{pvu}) / (n_{pvu} k_o)$$

$$n_{pvu} = \text{Real}$$

$$\partial\omega/\partial n_{pvu} < 0. \quad (6-5.2)$$

Secondly we consider those modes such that n_{pvu} will be purely imaginary.

$$(b1) \text{ If } jn_{pvu} > 0,$$

then for the mode which has axial refractive index n_{pef} such that

$$n_{\text{pef}} = -n_{\text{pvu}}$$

one gets

$$[2k_o / (\omega \mu_o)] \lambda_{\text{efl}} \iint_s \text{Re} \left\{ \left[\sum_j M_{jt} \cdot \delta_j \pi_j(n_{\text{pef}}) \right] \times \right. \\ \left. \begin{array}{l} z = 0 \\ \varphi = 0 \end{array} \right\}$$

$$\left[\sum_j K_{jt}^*(n_{\text{pef}}^*) \cdot \delta_j^*(n_{\text{pef}}^*) \nabla_t \pi_j^*(n_{\text{pef}}^*) \right] \} \cdot dS$$

$$+ [2T_e k_o \epsilon_o^2 \omega / (eN_o)] \lambda_{\text{efl}} \iint_s \text{Re} \left\{ \left[\sum_j R_j \delta_j \pi_j(n_{\text{pef}}) \right] \cdot \right. \\ \left. \begin{array}{l} z = 0 \\ \varphi = 0 \end{array} \right\}$$

$$\left[\sum_j V_{33}^*(n_{\text{pef}}^*) \cdot \delta_j^*(n_{\text{pef}}^*) \pi_j^*(n_{\text{pef}}^*) \right] \} \cdot dS = U(q, n_{\text{pef}}^*) \\ \begin{array}{l} z = 0 \\ \varphi = 0 \end{array} \cdot [1 / (n_{\text{pef}} k_o)]$$

where U is defined with Eq. (6-4,2),

$$j n_{\text{pef}} = \text{Real}$$

$$j n_{\text{pef}} < 0 \quad (6-6.1)$$

$$(b2) \text{ If } j n_{\text{pvu}} < 0$$

then for the mode which corresponds to n_{pvu} where

$$n_{\text{pef}} = -n_{\text{pvu}}$$

Eq. (6-4) yields to

$$- [2k_o / (\omega \mu_o)] \lambda_{ef} \iint_s \operatorname{Re} \left\{ \left[\sum_j M_{jt} \cdot \delta_j \pi_j (n_{pef}) \right] \times \left[\sum_j K_{jt}^* (n_{pef}^*) \cdot \delta_j^* (n_{pef}^*) \nabla_t \pi_j^* (n_{pef}^*) \right] \right\} . dS$$

$z = 0$
 $\varphi = 0$

$z = 0$
 $\varphi = 0$

$$- [2T_e k_o \epsilon_o^2 \omega / (e N_o)] \lambda_{ef} \iint_s \operatorname{Re} \left\{ \left[\sum_j R_j \delta_j \pi_j (n_{pef}) \right] \left[\sum_j V_{3j}^* (n_{pef}^*) \delta_j^* (n_{pef}^*) \pi_j^* (n_{pef}^*) \right] \right\} . dS$$

$z = 0$
 $\varphi = 0$

$z = 0$
 $\varphi = 0$

$$= U(q, n_{pef}^*) / (n_{pef} k_o) \quad (6-6.2)$$

$$j n_{pef} = \text{Real}$$

$$j n_{pef} > 0. \quad (6-6.2)$$

Eqs. (6-5) and (6-6) determine the magnitudes of the fields of different modes for a given field of sources " " inside the waveguide. These two equations can be combined in one expression as

$$(\lambda_{mk})^2 W_o(mk) = \psi(\tilde{J}, \tilde{K}, \tilde{\beta}, \tilde{F}, mk) \quad (6-7.1)$$

where the function W_o can be defined by the equation

$$W_o = \operatorname{Re} (W_1 + W_2) \quad (6-7.2)$$

where

$$W_1(mk) = [k_o / (\omega \mu_o)] (-1)^{k-1}$$

$$\iint_s \left\{ \left[\sum_j M_{jt} \cdot \delta_j \nabla_t \pi_j (n_{pml}) \right] \times \left[\sum_j K_{jt}^* (n_{pml}^*) \delta_j^* (n_{pml}^*) \pi_j^* (n_{pml}^*) \right] \right\} . dS$$

$z = 0$
 $\varphi = 0$

$z = 0$
 $\varphi = 0$

(6-7.3)

$$W_2(m\ell k) = [T_e k_o \epsilon_o^2 \omega / (eN_o)] (-1)^{k-1}$$

$$\iint_S \left[\sum_j R_j \delta_j \pi_j(n_{pm\ell}) \right] \cdot \left[\sum_j V_{33j}^* (n_{pm\ell}^*) \delta_j^* (n_{pm\ell}^*) \pi_j^* (n_{pm\ell}^*) \right] dS \quad (6-7.4)$$

$z = 0$
 $\varphi = 0$

$$\psi(\underline{J}, \underline{K}, \underline{\beta}, \underline{I}, m\ell) = U(\underline{J}, \underline{K}, \underline{\beta}, F, n_{pm\ell}^*) / (n_{pm\ell} k_o) \quad (6-7.5)$$

and $d\tilde{S}$ has to be taken as

$$d\tilde{S} = dS_z^{\wedge} \quad (6-7.6)$$

For a waveguide which is short circuited at one end, say at $z = 0$, similar expressions as Eq. (6-7) can be derived defining a field F_n^1 instead of the field F_n which we have been using, by assuming the z dependence to be $-2j \sin(k_o n_{pvu} z)$ where

$$n_{pvu} = n_{pm\ell}^*$$

In this case, the left hand side of Eq. (6-7.1) remains unchanged, whereas on the right hand side of Eq. (6-7.1) and hence on the right hand side of Eq. (6-7.5), the factor " $[\exp(-jk_o n_{pm\ell}^* z)]$ " has to be replaced with the factor " $[-2j \sin(k_o n_{pm\ell}^* z)]^*$ ".

The power flowing through the waveguide is given by

$$P + jQ = (1/2) \iint_S [\underline{E} \times \underline{H}^* - T_e (N/N_o) \underline{I}^*] \cdot d\underline{S}. \quad (6-8)$$

Inserting the expanded expression

$$\underline{F} = \sum_{m,l} \lambda_{ml} \underline{F}_{ml}$$

where \underline{F} can be \underline{E} , \underline{H} , \underline{N} or \underline{I} and (ml) represents mode index, one gets

$$P + jQ = (1/2) \iint_S \left[\sum_{ml} \sum_{pq} \lambda_{ml} \lambda_{pq}^* (\underline{E}_{ml} \times \underline{H}_{pq}^* - T_e (N_{ml}/N_o) \underline{I}_{pq}^*) \right] \cdot d\underline{S}.$$

Using the orthogonality relations (4-13), Eq. (6-9) can be reduced to

$$P + jQ = (1/2) \sum_{m,l} |\lambda_{ml}|^2 \iint_S [\underline{E}_{ml} \times \underline{H}_{ml}^* - T_e (N_{ml}/N_o) \underline{I}_{ml}^*] \cdot d\underline{S} \quad (6-10)$$

which is valid only for the waveguide with its axis parallel to the dc magnetic field. This result shows that if the waveguide modes did not have the orthogonality property which we made use of, the total power flow could not be the sum of the power flow for each mode.

The real part P of the Eq. (6-10) is due to all the propagating modes, i.e. with real n_p and the imaginary part Q is due to all the attenuating modes, i.e. modes with imaginary n_p .

With Eq. (6-7.1) it is indicated that

$$\iint_{S_k} [E_{m\ell} \times H_{m\ell}^* - T_e (N_{m\ell}/N_o) I_{m\ell}^*] \cdot dS = k_o n_{pm\ell} (W_1 + W_2)$$

Hence one can rewrite Eq. (6-10) as

$$P + jQ = (1/2) \sum_{m,\ell} |\lambda_{m\ell}|^2 k_o n_{pm\ell} (W_1 + W_2), \quad (6-11)$$

or

$$P + jQ = (1/2) \sum_{m,\ell} |\psi(m\ell)|^2 k_o n_{pm\ell} (W_1 + W_2) / (2W_o)^2. \quad (6-12)$$

Let I_{input} be the input current to a probe in the guide, then its input impedance is given by

$$R + jX = [1/|I_{input}|^2] \sum_{k=1}^2 \sum_{m,\ell} |\psi(m\ell)|^2 k_o n_{pm\ell} (W_1 + W_2) / |2W_o(m\ell k)|^2. \quad (6-13)$$

The expression for W_i can be given as

$$W_i = (-1)^k [2\pi / (\omega \mu_o)] k_o^{14} G_i \quad (i = 1, 2) \quad (6-14.1)$$

where G_1 and G_2 are dimensionless as given in the following form

$$G_i = \int_{\rho=0}^{k_o r_o} \mathcal{X}_i d\rho \quad (i = 1, 2) \quad (6-14.2)$$

For the cold plasma case,

$$\begin{aligned}
 \mathcal{I} = & (L_2^1)^2 J_m^2(n_{t2} k_o r_o) \left\{ \begin{aligned} & 2m[(m/p) J_m^2(n_{t1} \rho) - n_{t1} J_m(n_{t1} \rho) J_{m+1}(n_{t1} \rho)] \\ & [n_p^2 |K_{xy}|^2 + (L_1^1 + n_p^2 S_1^1) S_1^1 \\ & -j K_{xy} (L_1^1 + 2n_p^2 S_1^1)] \\ & + p n_{t1}^2 J_{m+1}^2(n_{t1} \rho) [n_p^2 |K_{xy}|^2 + (L_1^1 + n_p^2 S_1^1) S_1^1] \end{aligned} \right\} \\
 & + (L_1^1)^2 J_m^2(n_{t1} k_o r_o) \left\{ \begin{aligned} & 2m[(m/p) J_m^2(n_{t2} \rho) - n_{t2} J_m(n_{t2} \rho) J_{m+1}(n_{t2} \rho)] \\ & [n_p^2 |K_{xy}|^2 + (L_2^1 + n_p^2 S_2^1) S_2^1 \\ & -j K_{xy} (L_2^1 + 2n_p^2 S_2^1)] \\ & + p n_{t2}^2 J_{m+1}^2(n_{t2} \rho) [n_p^2 |K_{xy}|^2 + (L_2^1 + n_p^2 S_2^1) S_2^1] \end{aligned} \right\} \\
 & + L_1^1 L_2^1 J_m(n_{t1} k_o r_o) J_m(n_{t2} k_o r_o) \left\{ \begin{aligned} & m[(2m/p) J_m(n_{t1} \rho) J_m(n_{t2} \rho) - n_{t1} J_{m+1}(n_{t1} \rho) J_m(n_{t2} \rho) \\ & - n_{t2} J_{m+1}(n_{t2} \rho) J_m(n_{t1} \rho)] \\ & [-2n_p^2 |K_{xy}|^2 - (L_1^1 + n_p^2 S_1^1) S_2^1 - (L_2^1 + n_p^2 S_2^1) S_1^1 \\ & + j n_p^2 K_{xy} (S_1^1 + S_2^1) - j (L_1^1 + n_p^2 S_1^1) K_{xy}^* \\ & - j (L_2^1 + n_p^2 S_2^1) K_{xy}^*] \end{aligned} \right\}
 \end{aligned}$$

$$+L_1^1 L_2^1 J_m(n_{t1} k_o r_o) J_m(n_{t2} k_o r_o) p_{n_{t1} n_{t2}} J_{m+1}(n_{t1}) J_{m+1}(n_{t2}),$$

$$[-2n_p^2 |k_{xy}|^2 - (L_1^1 + n_p^2 s_1^1) s_2^1 - (L_2^1 + n_p^2 s_2^1) s_1^1]$$

(6-15)

where

$$= k_o r$$

$$s_{1,2}^1 = s_{1,2}/k_o^2 \quad [s_{1,2} \text{ defined by Eq. (3-3)}]$$

$$L_{1,2}^1 = L_{1,2}/k_o^2 \quad [L_{1,2} \text{ defined by Eq. (3-4)}].$$

$$X_2 = 0.$$

Let the current distribution $\tilde{J}(r, \varphi, z)$ be given as

$$J(r, \varphi, z) = |I_{\text{input}}| \tilde{j}(r, \varphi, z) \quad (6-16.1)$$

Then, it can be shown that the expression for ψ will be found in the form of

$$\psi = k_o^6 I_{\text{input}} F(n_{pm\ell}) \quad (6-16.2)$$

where F is some dimensionless function of the sources.

Inserting Eqs. (6-14) and (6-16) into Eq. (6-13) one obtains

$$R + jX = [1/|I_{\text{input}}|^2] \sum_{k=1}^2 \sum_{m,\ell} (-1)^k (1/8\pi) (\mu_o/\epsilon_o)^{1/2} n_{pm\ell} (G_1 + G_2)$$

$$\{|F(n_{pm\ell})|^2 / [\text{Re}(G_1 + G_2)]^2\}. \quad (6-17)$$

As a specific example, if the current distribution $\underline{J}(r, \phi, z)$ in the waveguide is given as

$$\underline{J} = [I(z) \delta(r)/(2\pi)] \underline{z}^{\wedge} \quad (6-18.1)$$

one finds that

$$\psi(\underline{J}, m, l) = 0 \quad \text{for } m \neq 0$$

$$\psi(\underline{J}, 0, l) = -k_o n_{pol}^2 \int \sum_j M_j^*(n_{pol}^*) \delta_j^*(n_{pol}^*) \cdot \pi_j^*(n_{pol}^*) \exp(j k_o n_{pol} z) I(z) \cdot dz \quad (6-18.2)$$

$$\begin{matrix} r = 0 \\ \phi = 0 \\ z = 0 \end{matrix}$$

For the cold plasma filled waveguide, assuming

$$k_o r_o = 3 \quad (6-19.1)$$

$$I(z) = \begin{cases} I_{input} \cos(k_o n_p z) & \text{for } k_o n_p z \leq \pi/2 \\ 0 & \text{for } k_o n_p z \geq \pi/2 \end{cases} \quad (6-19.2)$$

$$X = .2 \quad (6-19.3)$$

$$Y = .2 \quad (6-19.4)$$

and assuming that the waveguide is short circuited at $z = 0$, the real part R of the input impedance is found to be

$$R = 21.3 (\mu_o / \epsilon_o)^{1/2}. \quad (6-19.5)$$

However, the waveguide has infinite number of attenuating modes, therefore, the computation of the reactance X requires the summation of an infinite series.

VII. ANISOTROPIC COLD PLASMA FILLED WAVEGUIDES WITH THE dc MAGNETIC FIELD OBLIQUE TO THE GUIDE AXIS

Let the guide axis z' make an angle α with the dc magnetic field \underline{B}_0 which is parallel to z axis. Without the loss of generality one may assume that \underline{B}_0 lies in the $x'z'$ plane and consider that the field inside the waveguide is composed of plane waves of which the propagation constants are represented by points on the refractive index surfaces [Figure (7-1)].

With respect to the principal coordinate system the direction cosines of \underline{B}_0 will be $\sin\alpha$, 0, and $\cos\alpha$.

If we fix the azimuthal and axial coordinates of a propagation vector \underline{n} as θ and n_p according to the coordinate axes x' , y' , z' , then the radial component n_r of this vector can be determined as a multi-valued function of n_p and θ . The direction cosines of \underline{n} are then $n_r \cos\theta/n$, $n_r \sin\theta/n$ and n_p/n where $n_r^2 + n_p^2 = n^2$.

Forming the scalar product of \underline{B}_0 and \underline{n} one can find the angle β between \underline{B}_0 and \underline{n} as follows:

$$(n_r^2 + n_p^2) \cos^2 \beta = (n_r \cos\theta \sin\alpha + n_p \cos\alpha)^2. \quad (7-1)$$

The dispersion relation between n_t and n_p can now be given below as a function of the medium parameters K_0 , K_+ , K_- , K^1 and K_1 , all of which can be expressed in terms of X , Y and Z :

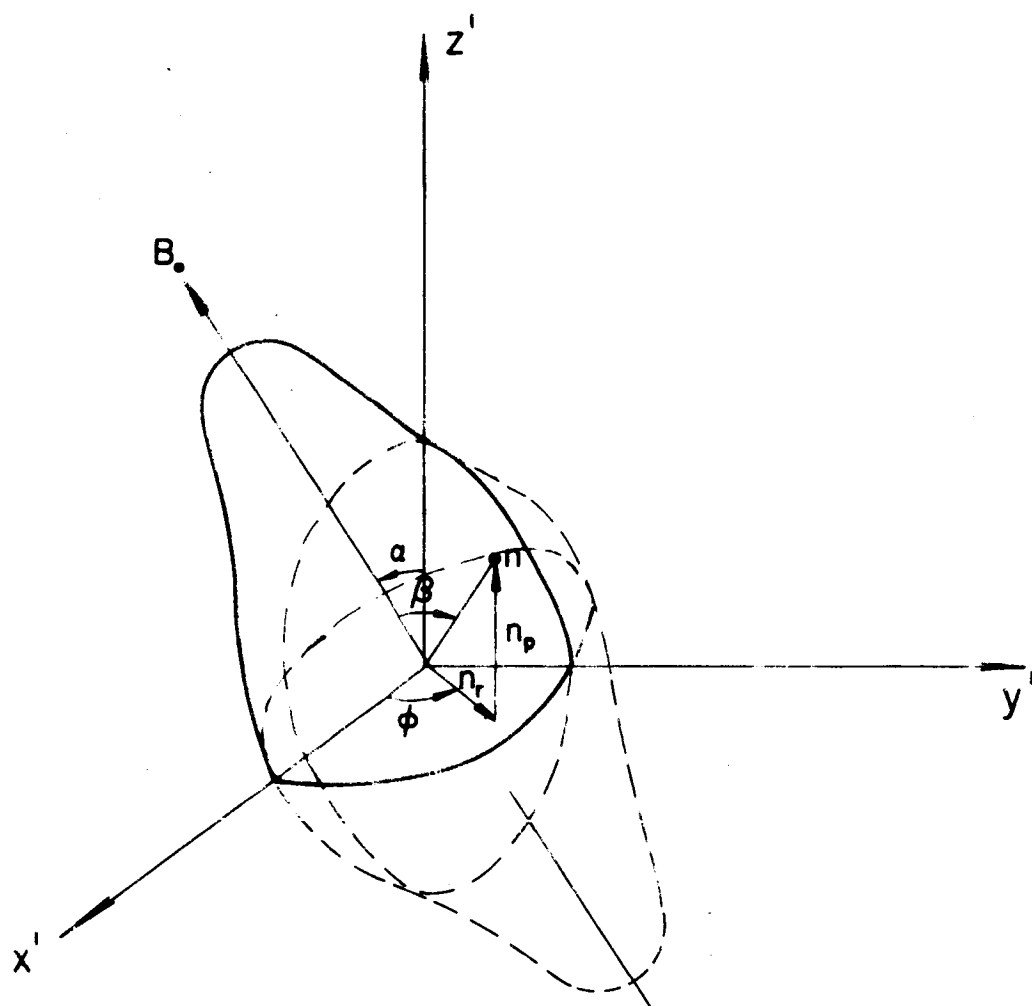


Figure (7-1) One of the index surfaces of the cold plasma where the dc magnetic field makes an angle α with the guide axis z' .

$$\begin{aligned}
& K_0 (n_r^2 + n_p^2 - K_+) (n_r^2 + n_p^2 - K_-) \cos^2 \beta \\
& + K^1 (n_r^2 + n_p^2 - K_0) (n_r^2 + n_p^2 - K_1) (1 - \cos^2 \beta) = 0
\end{aligned} \tag{7-2}$$

where

$$K_0 = 1 - X / (1 - jZ)$$

$$K_+ = 1 - X / (1 + Y - jZ)$$

$$K_- = 1 - X / (1 - Y - jZ)$$

$$K_1 = 2 / [(1/K_+) + (1/K_-)]$$

$$K^1 = 1 - X(1 - jZ) / [(1 - jZ)^2 - Y^2]$$

Solving $\cos^2 \beta$ from Eq. (7-1) and inserting it into Eq. (7-2)

one obtains

$$\begin{aligned}
& K_0 (n_r^2 + n_p^2 - K_+) (n_r^2 + n_p^2 - K_-) (n_r \cos \theta \sin \alpha + n_p \cos \alpha)^2 \\
& + K^1 (n_r^2 + n_p^2 - K_0) (n_r^2 + n_p^2 - K_1) [n_r^2 + n_p^2 - (n_r \cos \theta \sin \alpha + n_p \cos \alpha)^2] = 0
\end{aligned}$$

which is an algebraic equation of six degrees in n_r . Thus n_r is a six valued function of n_p and θ . For the uniaxial case this value reduces to four. With a reasoning analogous to that considered in Chapter III, Section 3, one can consider that the field inside the waveguide is composed of plane waves with transverse propagation vectors falling into intervals of angle $\theta < \theta_1 < \theta + d\theta$ where the magnitude of the wave can

be expressed in a Fourier series in terms of ϕ because of the periodicity in ϕ with a period of 2π ; i.e.,

$$A_o(\phi_1) = \sum_{m=-\infty}^{\infty} A_{mo} \exp(jm\phi_1). \quad (7-4)$$

Therefore, one can consider Figure (7-2) which is similar to Figure (3-1) and find the field at a point $P(r, \varphi, z^1)$ in the following form:

$$dE_{z^1}(n_p, m_1, \phi | r, \varphi, z^1) = A_{m_1 o} \exp(jm_1 \phi - jk_o n_p z^1) \cdot \exp[jk_o n_{r1}(n_p, \phi) r \sin \theta]. \quad (7-5)$$

Again inserting

$$\phi = \varphi - \theta + \pi/2 \quad (7-6)$$

one gets for $E_{z^1}(n_p, m_1/r, \varphi)$

$$E_{z^1}(m_1) = A_{m_1 o} \exp(jm_1 \varphi - jk_o n_p z^1 + jm_1 \pi/2) \cdot \int_{\theta=0}^{2\pi} \exp[jk_o n_{t1}(n_p, \varphi, \theta) r \sin \theta - m_1 \theta] d\theta. \quad (7-7)$$

To determine the other components of \underline{E} one can again write

$$dE_r(m_1) = \sin \theta dE_{x_1} - \cos \theta dE_{y_1} \quad (7-8.1)$$

$$dE_{\varphi}(m_1) = (\cos \theta) dE_{x_1} + (\sin \theta) dE_{y_1} \quad (7-8.2)$$

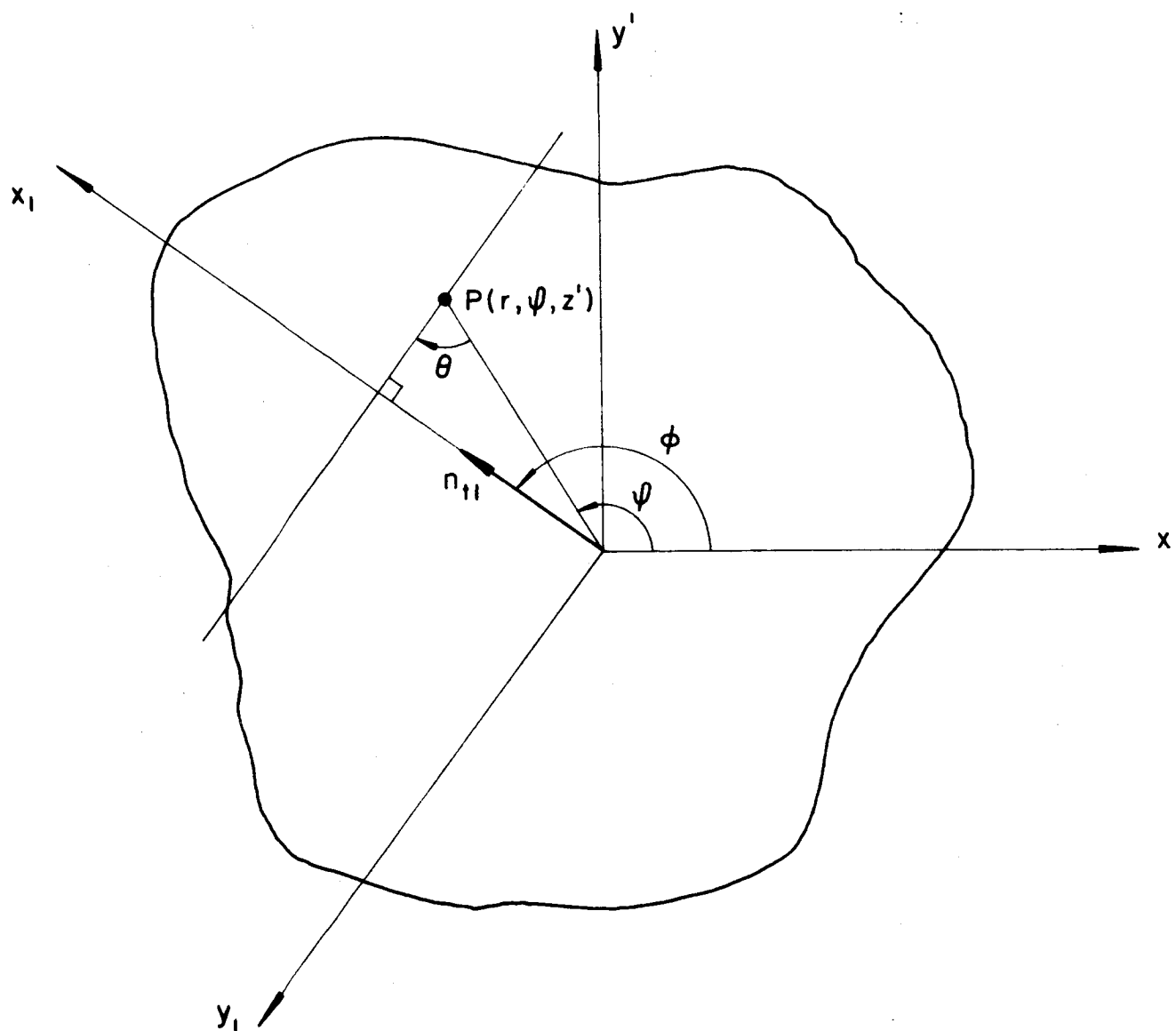


Figure (7-2) Contribution of a plane wave of transverse propagation index n_{r1} to the field at a point $P(r, \phi, z')$

In these expressions, E_{x_1} , E_{y_1} , and E_{z_1} are related to $E_{x'}$, $E_{y'}$, and $E_{z'}$ through the following equations:

$$\begin{bmatrix} E_{x_1} \\ E_{y_1} \\ E_{z_1} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_{x'} \\ E_{y'} \\ E_{z'} \end{bmatrix} \quad (7-9)$$

and

$$\begin{bmatrix} E_{x'} \\ E_{y'} \\ E_{z'} \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \quad (7-10)$$

By making use of Eq. (7-10) together with Eqs. (2-17.8) and (2-17.9) one obtains

$$E_{x'} = [\cos\alpha(C_1/C_3) + \sin\alpha]E_z$$

$$E_{z'} = [-\sin\alpha(C_1/C_3) + \cos\alpha]E_z$$

or

$$E_{x'} = [(C_1 \cos\alpha + C_3 \sin\alpha)/(-C_1 \sin\alpha + C_3 \cos\alpha)]E_{z'} \quad (7-10.1)$$

and similarly

$$E_{y_1} = [C_2 / (-C_1 \sin \alpha + C_3 \cos \alpha)] E_{z_1} \quad (7-10.2)$$

Inserting Eqs. (7-10) into Eq. (7-9) one obtains

$$E_{x_1} = \{[(C_1 \cos \alpha + C_3 \sin \alpha) \cos \theta + C_2 \sin \theta] / (-C_1 \sin \alpha + C_3 \cos \alpha)\} E_{z_1} \quad (7-11.1)$$

$$E_{y_1} = \{[-(C_1 \cos \alpha + C_3 \sin \alpha) \sin \theta + C_2 \cos \theta] / (-C_1 \sin \alpha + C_3 \cos \alpha)\} E_{z_1} \quad (7-11.2)$$

C_1 , C_2 and C_3 are now to be determined from Eqs. (2-18.1), (2-18.8) and (2-18.9) by inserting the following equivalent expressions for the operators:

$$d_x = -jk_o n_x \quad (7-12.1)$$

$$d_y = -jk_o n_y \quad (7-12.2)$$

$$d_z = -jk_o n_z \quad (7-12.3)$$

where

$$\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} n_{x_1} \\ n_{y_1} \\ n_{z_1} \end{bmatrix}$$

and

$$n_{x1} = n_r \cos\theta = -n_r \sin(\varphi - \theta)$$

$$n_{y1} = n_r \sin\theta = n_r \cos(\varphi - \theta)$$

$$n_{z1} = n_p$$

These relations lead to the following

$$n_x = -n_r \cos\alpha \sin(\varphi - \theta) - n_p \sin\alpha \quad (7-13.1)$$

$$n_y = n_r \cos(\varphi - \theta) \quad (7-13.2)$$

$$n_z = -n_r \sin\alpha \sin(\varphi - \theta) + n_p \cos\alpha \quad (7-13.3)$$

After having found the expressions for E_r and E_φ , one can apply the boundary conditions as given by Eqs. (3-8) and find the characteristic equation for the waveguide. However, Eq. (7-3), being a polynomial of sixth degree in n_r one cannot derive an explicit expression for n_r .

If dc magnetic field is infinitely strong, then K_+ , K_- , K^1 , and K_1 all become equal to 1, and, therefore, the equation reduces to a polynomial of fourth degree

$$\begin{aligned} & K_0 (n_r^2 + n_p^2 - 1) (n_r \cos\theta \sin\alpha + n_p \cos\alpha)^2 \\ & + (n_r^2 + n_p^2 - K_0) [n_r^2 + n_p^2 - (n_r \cos\theta \sin\alpha + n_p \cos\alpha)^2] = 0 \end{aligned} \quad (7-14)$$

from which one can derive an explicit expression of n_r in terms of n_p and ϕ .

Moreover, since in Eq. (7-7) n_{r1} is a function of ϕ , $E_{z1}(m_1)$ is not represented by a single Fourier component in ϕ . One can find the Fourier series expansion of the total axial component of \vec{E} by first summing up E_{z1} 's for possible m_1 's and then finding the Fourier component, say the ℓ 'th, by multiplying the summation by $e^{j\ell\phi}$ and integrating over ϕ . This introduces an infinite set of terms into the expressions of the field components and hence into the boundary conditions. One can only say that, in case the dc magnetic field is not parallel to the waveguide axis one can no longer have simple modes such as the ones one has when the dc magnetic field is parallel to the wave axis.

VIII. CONCLUSIONS

Waves in a linearized homogeneous plasma of unbounded region are investigated. In order to evaluate the effects of electron temperature and resonance on the wave propagation, three plasma models in increasing complexity are considered: the incompressible (or the cold) plasma, the compressible (or the warm) plasma, and the microscopic model. The first model is the usual one by treating the plasma as a dielectric; the second is based on the transport equation; and the third is based on the Boltzmann equation with assumed collision integral and velocity distribution at equilibrium.

It is found that the compressibility of the plasma introduces some modifications to the refractive index surfaces. First there exist three surfaces instead of two as in the cold plasma model. The smallest surface differs little from that of the cold plasma, only by a quantity dependent on the fourth power of the ratio of the acoustic speed to the light speed in free space. Parts of the other two surfaces correspond to the second surface of the cold plasma since they degenerate into the latter as the electron temperature approaches zero. Now it is found that the surface can extend to infinity only when $Y^2 \geq 1$, in contrast to the cold plasma case. In fact for this case the asymptotes are at angles (measured from the magnetizing field) larger than those of the cold plasma model. In the neighborhood of longitudinal direction the refractive index of the third surface is real and nearly equal to that of the cold plasma. In the neighborhood of transverse direction the

refractive index of the second surface is purely imaginary and nearly equal to that of the cold plasma in the same region.

The effect of the electron velocity distribution on the refractive indices is analyzed, based on the aforementioned Boltzmann equation approach. Because of the complexity, only the propagation in two interesting angular regions, namely along and transverse to the static magnetizing field are considered. It is found that for propagation in the general longitudinal direction the refractive index cannot be a large real number in contrast to the results of the warm plasma model near gyroresonance. Furthermore it is found that from the Boltzmann equation approach the refractive index in the transverse direction becomes infinity at $Y = 1/p$, $p = \text{an integer}$.

Waves in a circular waveguide filled with either cold or warm magnetoplasma are studied. When the magnetizing field is parallel to the guide axis the modal waves can be expressed in terms of known functions, using the boundary condition that on the guide wall the tangential electric field vanishes, and for the warm plasma the normal electron velocity also vanishes. The characteristic equations for both plasma models are numerically evaluated and compared. It is found that the longitudinal propagation constants of the warm plasma model consists of two types. The first type can be identified as those of the cold plasma but slightly perturbed by a quantity dependent on the ratio of the acoustic to the light speeds in free space. However, some field components associated with these modes (E_p, H_ϕ) differ substantially for the two plasma models. Modes of the other type arises from the

compressibility of the plasma and the additional boundary condition on the normal electron velocity. The longitudinal propagation constant of the second type change rapidly with the guide radius in wavelength and are so densely located in the Brillouin diagram that they resemble a continuous spectrum.

For a given source inside the waveguide orthogonality relations between the modal solutions are used to determine the relative power distributed among various modes. Hence, the real part of the impedance of an antenna placed in the guide can be computed. However, as stated above, in general, the fields for the cold and the warm plasma models differ even if they belong to the modes of nearly equal propagation constants. Moreover, the warm plasma model brings forth additional modes. Therefore, it is expected that the antenna impedance will be different for these two models.

The study of an anisotropic guide with an oblique magnetizing field shows that the modal solutions cannot be expressed in terms of known functions, but solutions in terms of series expansions of elementary functions may be used. However, even for uniaxial cold plasma, the expressions become very involved.

APPENDIX I

ENERGY RELATIONS AND RESTRICTIONS ON THE MATRIX (K)

In part I we have considered three plasma models: the cold plasma, the warm plasma based upon the transport theory or the fluid model, and the warm plasma based upon Boltzmann theory. The discussion of all these three models can begin with the assumption of a one particle distribution function f for every species of the ions of the plasma, which, in the most general case can be a function of the velocity and position of the particle and time. For the cold plasma case, however, f reduces to a δ function of the velocity vector as long as the medium is not perturbed.

For the distribution function f the Boltzmann equation should be satisfied

$$d_t f + \sum_{i=1}^3 v_i d_{x_i} f + \sum_{i=1}^3 (F_i/m) d_{v_i} f = (\partial f / \partial t)_{\text{collisions}} \quad (\text{A1-1})$$

where v_i is the i^{th} component of the position vector \underline{v} of the particle,*

x_i is the i^{th} component of the position vector \underline{r} of the particle,

F_i is the i^{th} component of the force acting on the particle,

m is the mass of the particle,

$(\partial f / \partial t)_{\text{collisions}}$ is the collision integral.

*In this appendix the letter v is used for the velocity of individual ions. However, in the main text the letter \tilde{v} is used for the fluid model warm plasma to represent the average velocity of ions as it will be defined by Eq. (A1-2.3)

Multiplying both sides of Eq. (A1-1) with a function of \underline{v} , which for example can be expressed as $Q(\underline{v})$ and integrating over the whole \underline{v} space one can derive the following equation: (2)

$$\begin{aligned} d_t(N\bar{Q}) + \sum_{i=1}^3 d_{x_i}(N\bar{v}_i Q) - \sum_{i=1}^3 \overline{NF_i d_{v_i} Q} \\ = \int_{-\infty}^{\infty} (\partial f / \partial t)_{\text{coll}} Q d\underline{v} \end{aligned} \quad (\text{A1-2.1})$$

where* $N = \int_{-\infty}^{\infty} f d\underline{v}$ (A1-2.2)

$$\overline{N\psi(\underline{v})} = \int_{-\infty}^{\infty} f \psi(\underline{v}) d\underline{v} \quad (\text{A1-2.3})$$

This equation is obtained with the assumptions that F_i is not**a function of v_i and as \underline{v} approaches to infinity $f(\underline{v})$ approaches to zero with a sufficient order of \underline{v} such that

$$\int_{-\infty}^{\infty} d_{v_i} (f F_i Q) d\underline{v} = 0.$$

By first taking $Q = 1$ and second $Q = \underline{v}$ one gets the first and the second Boltzmann transport equations which are also called the first and second moment equations.

$$d_t N_j + \nabla \cdot (N \bar{v}_j) = 0 \quad (\text{A1-3})$$

*In the main text, for fluid model warm plasma, for convenience, the letter \underline{v} is used for $\underline{\bar{v}}$.

**Although in the plasma we consider $\underline{F} = \underline{F}(\underline{v})$ still $F_i \neq F_i(\underline{v})$, because we have

$$\underline{F} = -e(\underline{v} \times \underline{B}_0).$$

and

$$d_t (N_j m_j \bar{\mathbf{v}}_j) + \sum_{i=1}^3 d_{x_i} (N_j m_j \bar{\mathbf{v}}_i \bar{\mathbf{v}}_j) - N_j \sum_{i=1}^3 \bar{\mathbf{F}}_i x_i = \int_{-\infty}^{+\infty} (\partial f / \partial t)_{\text{coll}} m_j \bar{\mathbf{v}}_j d\mathbf{v} \quad (\text{A1-4})$$

The Maxwell equation for the Curl of \mathbf{H} will now take the form

$$\nabla \times \mathbf{H} = \epsilon_0 d_t \mathbf{E} + \sum_j \mathbf{I}_j + \mathbf{J} \quad (\text{A1-5})$$

where $\mathbf{I}_j = q_j N_j \bar{\mathbf{v}}_j$

q_j is the charge of the j and species of ion and \mathbf{J} represent the electric source current.

The second Maxwell Equation remains unchanged, namely

$$\nabla \times \mathbf{E} = -\mu_0 d_t \mathbf{H} - \mathbf{K} \quad (\text{A1-6})$$

where \mathbf{K} represents the magnetic source current.

Assume that the applied uniform static magnetic field \mathbf{B}_0 is sufficiently strong so that the following linearized relation is valid:

$$\sum_i \bar{\mathbf{F}}_i x_i = q_i (\mathbf{E} + \bar{\mathbf{v}}_i \times \mathbf{B}_0).$$

Dot multiplying Eqs. (A1-3), (A1-4), (A1-5) and (A1-6) with $k_B T_j N_j / N_{oj}$, $N_j \bar{\mathbf{v}}_j / q_j$, $-\mathbf{E}$ and \mathbf{H} , respectively and adding the four equations one finds (2)

$$\nabla \cdot [\mathbf{E} \times \mathbf{H} + \sum_j ((k_B T_j N_j / N_{oj}) \bar{\mathbf{v}}_j)] = -d_t [(1/2) \mu_0 H^2 + (1/2) \epsilon_0 E^2 + \sum_j (1/2) N_{oj} m_j \bar{\mathbf{v}}_j^2 + \sum_j (1/2) (k_B / N_{oj}) T_j \bar{N}_j^2]$$

$$-\tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}} - \tilde{\mathbf{K}} \cdot \tilde{\mathbf{H}}^* \quad (\text{A1-7})$$

where N_{oj} is the equilibrium value of N . For a source free medium, the last two terms of the right hand side of the above equation drop.*

For a cold plasma, inserting $T = 0$ one finds the conservation equation of the electromagnetic waves, which is the expression of the Poynting theorem. In that case the terms $\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}$, $(1/2)\mu_0 \tilde{\mathbf{H}}^2$, $(1/2)\epsilon_0 \tilde{\mathbf{E}}^2$ and $\sum_j (1/2)N_{oj}m_j \tilde{\mathbf{v}}_j^2$ represent the density of power flow, the stored magnetic energy density, the stored electric energy density and the kinetic energy density gained by the unit volume of the fluid of ions, respectively. On the other hand, for a medium where there are no electromagnetic fields, i.e., $\tilde{\mathbf{E}} = 0$ and $\tilde{\mathbf{H}} = 0$, Eq. (A1-7) reduces to the expression of the conservation theorem for acoustic disturbance of the ion gas. For that case the terms $\sum_j (\ell k T_j N_j \tilde{\mathbf{v}}_j)$, $\sum_j (1/2) N_{oj} m_j \tilde{\mathbf{v}}_j^2$ and $\sum_j (1/2) (\ell k / N_{oj}) T_j N_j^2$ would represent the density of energy flow due to the pressure gradient, the kinetic energy density and the thermal energy density of the compressible fluid, respectively. In the presence

*For a more general case one can add to Eq. (A1-3) a source term which creates particles and to Eq. (A1-4) a source term which represents forces applied on the particles other than the electromagnetic forces, and thus have a modified form of Eq. (A1-7). For the purpose of this text, however, this is unnecessary.

of the electromagnetic field, and temperature, the terms on the right hand side of Eq. (A1-7) would have the same meanings as explained above. The left hand side, which represents the total energy flow density can be called the modified or generalized Poynting vector.

In a monochromatic wave, by using the complex notation, Eq. (A1-7) can be written as follows:

$$\begin{aligned}
 (1/2) \nabla \cdot [\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* - \sum_j (\mathbf{k}_j T_j / N_{oj}) N_j \tilde{\mathbf{T}}_j^* / q_j] = \\
 (1/2) \tilde{\mathbf{H}}^* \cdot (-j\omega \mu_0 \tilde{\mathbf{H}}) \\
 - (1/2) \tilde{\mathbf{E}} \cdot (-j\omega \epsilon_0 \tilde{\mathbf{E}}^* + \sum_j q_j N_{oj} \tilde{\mathbf{V}}_j^*) \\
 - (1/2) \sum_j (\mathbf{k}_j T_j / N_{oj}) N_j (j\omega \mathbf{k}_j \cdot \tilde{\mathbf{N}}_j^*) \\
 + (1/2) \sum_j N_{oj} \tilde{\mathbf{V}}_j^* [-j\omega m_j \tilde{\mathbf{V}}_j + q_j \tilde{\mathbf{E}}_j + q_j (\tilde{\mathbf{V}}_j \times \tilde{\mathbf{B}}_0)] \\
 - (1/2) \tilde{\mathbf{J}}^* \cdot \tilde{\mathbf{E}} - (1/2) \tilde{\mathbf{K}} \cdot \tilde{\mathbf{H}}^*. \quad (A1-8)
 \end{aligned}$$

In the right hand side of Eq. (A1-8) the real part of the second term is the energy lost due to the electric field and current flow and the real part of the fourth term is the energy lost due to the motion of particles in the presence of a pressure gradient. Accordingly, the sum of the second and the fourth terms will be the total energy

loss due to mechanic and electric forces, the terms $\underline{\underline{E}} \cdot \underline{\underline{q}}_j N_{oj} \underline{\underline{v}}_j^*$ cancelling each other showing energy transformation from mechanical form to electrical form or vice versa. Since we assumed that there are no collisions and hence no energy contribution from the mechanical side to be consumed as ohmic losses, the second term must be purely imaginary. In case of a plane wave, where one can use a diadic $\underline{\underline{\epsilon}}$, then, one must have

$$\text{Re}[\underline{\underline{E}} \cdot (j\omega \underline{\underline{\epsilon}} \underline{\underline{E}})^*] \equiv 0. \quad (\text{A1-9})$$

From this one finds

$$\underline{\underline{E}} \cdot (-j\omega \underline{\underline{\epsilon}}^*) \cdot \underline{\underline{E}}^* + \underline{\underline{E}} \cdot j\omega \underline{\underline{\epsilon}}^T \underline{\underline{E}}^* = 0$$

or

$$\underline{\underline{E}} \cdot (-j\omega) (\underline{\underline{\epsilon}}^* - \underline{\underline{\epsilon}}^T) \cdot \underline{\underline{E}}^* \equiv 0.$$

This requires

$$\underline{\underline{\epsilon}}^* - \underline{\underline{\epsilon}}^T = 0$$

or

$$\underline{\underline{\epsilon}} = (\underline{\underline{\epsilon}}^T)^*.$$

APPENDIX II

THE DERIVATION OF THE MATRIX (K) ACCORDING TO THE
BOLTZMANN EQUATION APPROACH

Let $\sigma_{rij}^{'}$ denote the (ij)th element of the matrix $(\sigma_r^{'})$

where i and j can be 1 or 2 or 3. Using the transformation

$$\Phi = x + y \quad (A1-1)$$

$$\Phi' = -x + y \quad (A1-2)$$

form Eqs. (2-54.3) and (2-54.4) one attains that^{(3)*}

$$\sigma_{r11}^{'} = -[2e^2N/(m\omega_H)](1+\sin^2\theta\partial/\partial\sin^2\theta)I_{12} \quad (A1-3.1)$$

$$\sigma_{r12}^{'} = \sigma_{r21}^{'} = -[2e^2N/(m\omega_H)]\sin^2\theta(\partial/\partial\theta)I_3 \quad (A1-3.2)$$

$$\sigma_{r13}^{'} = \sigma_{r31}^{'} = \sigma_{r23}^{'} = \sigma_{r32}^{'} = [Ne^2/(m\omega)][k_o^2n^2v_T^2/(2\omega_H^2)](\sin^2\theta)I_{22} \quad (A1-3.3)$$

$$\sigma_{r33}^{'} = 2[e^2N/(m\omega_H)][1+(2n_p^2/v_T)(\partial/\partial n_p^2)]I_3 \quad (A1-3.4)$$

where

$$I_{12} = \int_0^\infty \exp\{-j2x[(\omega-j\nu)/\omega_H+1]\}\exp[-2(v_Tk_on/\omega_H)^2(x^2\cos^2\theta+\sin^2x\sin^2\theta)]dx \quad (A1-4)$$

$$I_{22} = \int_0^\infty \exp\{-j2x[(\omega-j\nu)/\omega_H+1/2]2x\}\exp[-2(v_Tk_on/\omega_H)^2(x^2\cos^2\theta+\sin^2x\sin^2\theta)]dx \quad (A1-5)$$

*In this section I_j is used to indicate the integrals only.

$$I_3 = \int_0^{\infty} \exp[-j2x(\omega - j\nu)/\omega_H] \exp[-2(v_T k_0 n / \omega_H)^2 (x^2 \cos^2 \theta + \sin^2 x \sin^2 \theta)] dx \quad (A1-6)$$

$$\text{where } v_T = (kT/m)^{1/2}, \quad (A1-7)$$

$$\text{Let } v_{11} = v + j\omega_H \quad (A1-8.1)$$

$$\text{and } v_{21} = v + j(1/2)\omega_H \quad (A1-8.2)$$

then one can write

$$I_3 = I_3(\theta, \nu) \quad (A1-8.3)$$

$$I_{11} = I_3(\theta, v_{11}) \quad (A1-8.4)$$

$$I_{21} = I_3(\theta, v_{21}) \quad (A1-8.5)$$

If one stipulates that

$$|k_0^2 n^2 kT / (m\omega_H^2)| \gg 1 \quad (2-56.1)$$

I_3 can be evaluated using the method of the steepest descent at points where the exponent of the integrand has saddle points in the complex plane.

If one defines:

$$f(z) = -(n^2/|n|^2)(\alpha z^2 + \beta \sin^2 z) \quad (A1-9)$$

$$\text{where } \alpha = \cos^2 \theta \quad (A1-10)$$

$$\text{and } \beta = \sin^2 \theta \quad (\text{A2-11})$$

the saddle points will be at the zeros of $d_z f(z)$ which is

$$d_z f(z) = -(n^2/|n|^2)[2\alpha z + \beta \sin 2z]. \quad (\text{A2-12})$$

Let

$$2z = x_1 + jy_1 = z_1 \quad (\text{A2-13})$$

then Eq. (A2-12) can be written as

$$\alpha x_1 + \beta \sin x_1 \cosh y_1 = 0 \quad (\text{A2-14.1})$$

$$\alpha y_1 + \beta \cos x_1 \sinh y_1 = 0. \quad (\text{A2-14.2})$$

Equations (A2-14) can be solved using graphical method. From Eqs.

(A2-14) one derives

$$x_1 / \tan x_1 = y_1 / \tanh y_1 \quad (\text{A2-15.1})$$

$$\text{and } -(\beta/\alpha) \sinh y_1 / y_1 = 1 / \cos x_1 \quad (\text{A2-15.2})$$

In Figure (A2-1) the curve corresponding to Eq. (A2-15.1) is plotted in $x_1 y_1$ plane. This curve is not periodic with respect to x_1 . The distances between the zeros of the curve and the odd multiples of $(\pi/2)$ decrease monotonically as (x_1) increases. The curves in Figure (A2-2) which are periodic with respect to x_1 represent the plot of Eq. (A2-15.2). The curves in Figures (A2-1) and (A2-2) all are symmetric with respect to the x_1 and y_1 axes. In Figure (A2-2.b) the

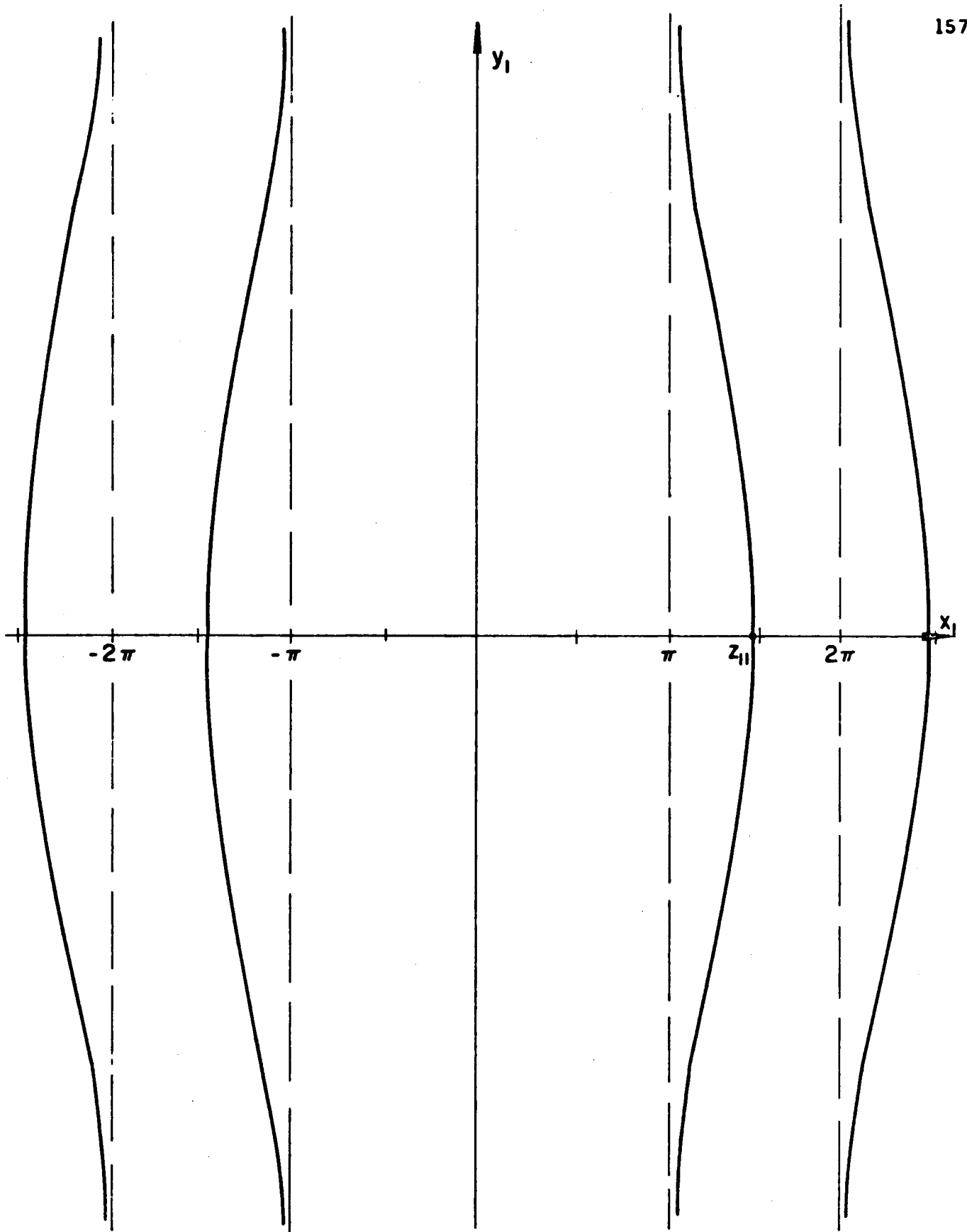


Figure (A2-1) The plot of $x_1 / \operatorname{tg} x_1 = y_1 / \operatorname{tgh} y_1$

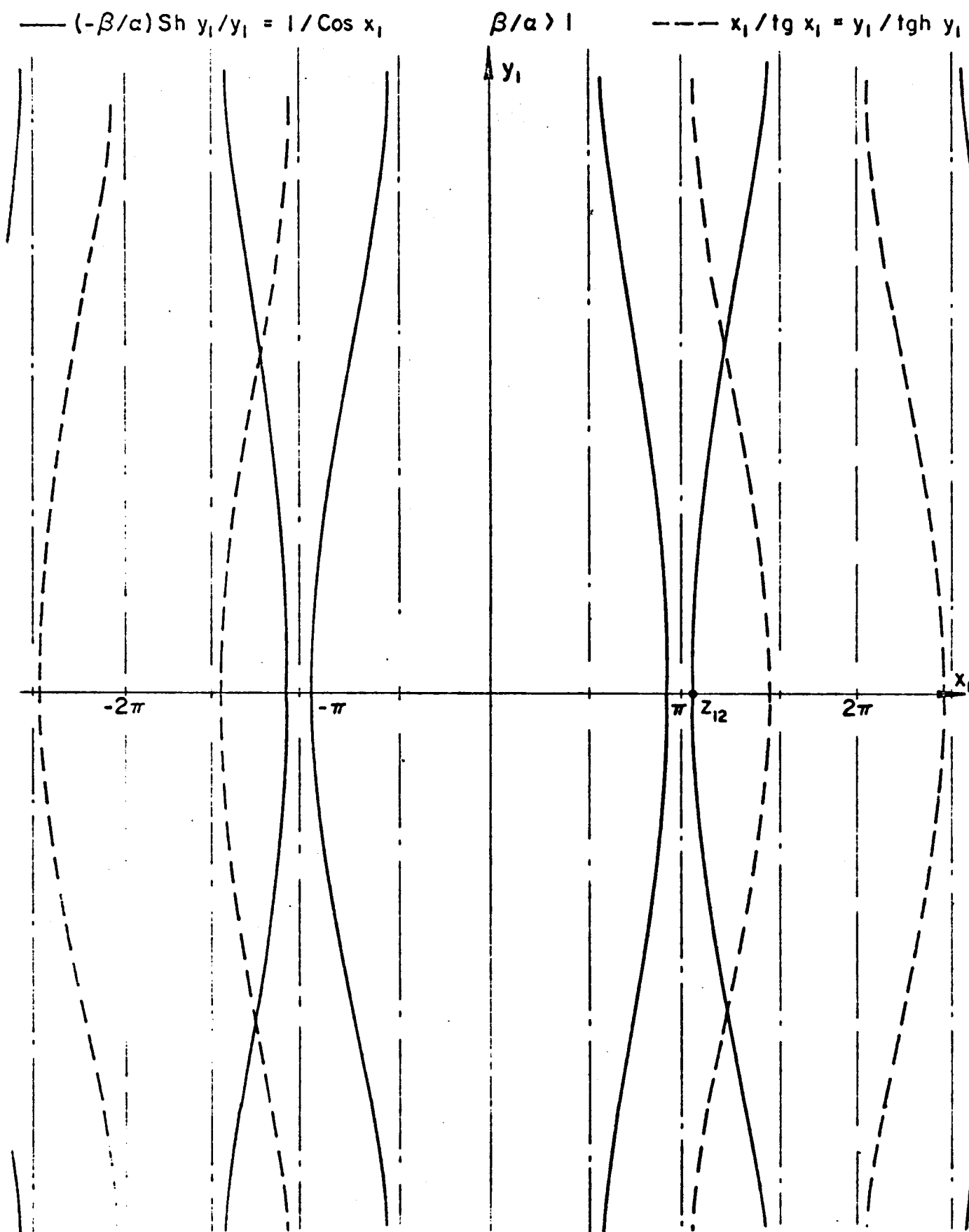


Figure (A2-2a) The plot of $(-\beta/\alpha) \text{Sh } y_1/y_1 = 1/\text{Cos } x_1$ for $\beta/\alpha > 1$.

— $(-\beta/\alpha) \operatorname{Sh} y_1 / y_1 = 1/\cos x_1$ $\beta/\alpha < 1$ - - - - $x_1 / \operatorname{tg} x_1 = y_1 / \operatorname{tgh} y_1$

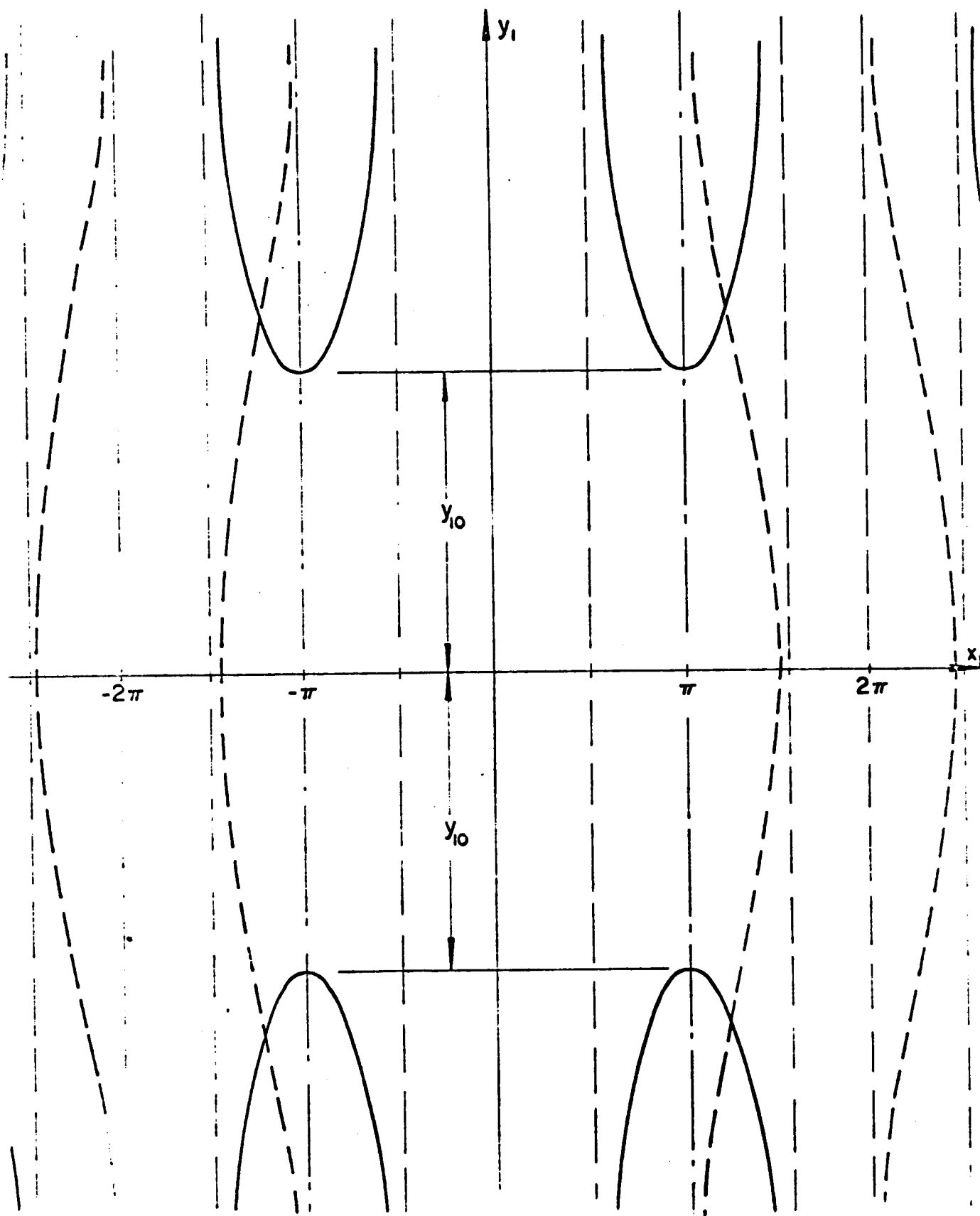


Figure (2-2b) The plot of $(-\beta/\alpha) \operatorname{Sh} y_1 / y_1 = 1/\cos x_1$ for $\beta/\alpha < 1$.

value of y_{10} is the solution of the equation

$$\sinh y_{10} / y_{10} = (\beta/\alpha)^{-1}$$

and it increases as (β/α) decreases and becomes arbitrarily large as (β/α) becomes arbitrarily small.

To find the roots of Eq. (A2-12) one should find the intersection of the curves plotted on Figures (A2-1) and (A2-2). One of the intersection points is (0,0). The other ones depend upon the values of the parameter (β/α) . For very small values of (β/α) they are complex conjugate pairs the abscissa of which are very close to multiples of π and the ordinate of which are very large. As β/α increases the deviations of their abscissa from these odd multiples of π increase and the absolute values of their ordinates decrease. For $\beta/\alpha = 4.61$ which corresponds to an angle $\theta = 65^\circ$, the first four roots which are closest to the y_1 axis become two real double roots and as β/α still increases we find four real roots moving on the real axis. As β/α still increases the next four complex roots closest to the y_1 axis become two real double roots and further they change to four real roots and the procedure continues in that manner. As β/α becomes arbitrarily large all the complex roots within same finite interval $(-x_1, +x_1)$ will become real roots within the same interval. Therefore, for $\theta = 90^\circ$ we will have infinitely many real roots and no complex roots.

For $\theta = 0$ or $(\beta/\alpha) = 0$ we have only one root at $x_1 = 0$, $y_1 = 0$. The contribution of this saddle point to the integral I_3 can be calculated as follows:

Let n be written as

$$n = |n| e^{j\phi}$$

then one has

$$f(z) = -z^2 \exp(j2\phi) \quad (\text{A2-16})$$

In the vicinity of

$$z_0 = 0$$

if one writes

$$z = r e^{j\psi}$$

where $r \ll 1$

one has for $f(z)$

$$f(z) = -r^2 \exp(j2\phi) \quad (\text{A2-17})$$

$$\text{or} \quad f(z) \approx -r^2 [\cos 2(\psi + \phi) + j \sin 2(\psi + \phi)] \quad (\text{A2-18})$$

Here for

$$d_r \{ \text{Re}[f(z)] \} = \min$$

one has to have

$$-\cos 2(\psi + \phi) = -1$$

or

$$\psi + \phi = P\pi$$

which implies

$$\psi = P\pi - \phi \quad (\text{A2-19})$$

Inserting $P = 0$ to Eq. (A2-19) one has for I_3

$$I_3 = (\pi)^{1/2} \exp(-j\phi) / (4t_1)^{1/2} \quad (\text{A2-20})$$

where $t_1 = 2v_T^2 k_o^2 |n|^2 / \omega_H^2$ (A2-21)

which implies

$$I_3 = (\pi/4t_1)^{1/2} \quad (A2-22)$$

where $t = 2v_T^2 k_o^2 n^2 / \omega_H^2$ (A2-23)

or $I_3 = (\pi)^{1/2} \omega_H / \sqrt{8} k_o n v_T$. (A2-24)

Since Eq. (A2-24) doesn't include v for $\theta = 0^\circ$, one can write

that

$$I_1 = I_2 = I_3 \quad \text{for } \theta = 0: \quad (A2-25)$$

Inserting the expression found for I_1 , I_2 and I_3 , one has

$$\sigma_r^{-1} = (s/n) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (A2-26)$$

When one increases the angle θ slightly from zero, one finds infinitely many saddle points as mentioned above. Let $x_o + jy_o = z_o$ be one of these saddle points. Then one has

$$\begin{aligned} -(\alpha z_o^2 + \beta \sin^2 z_o) &= -\alpha(x_o^2 - y_o^2) - \beta(\sin^2 x_o \cosh^2 y_o - \cos^2 x_o \sinh^2 y_o) \\ &\quad -j[2\alpha x_o y_o + \frac{1}{2} \sin 2x_o \sinh 2y_o] \end{aligned}$$

Under the condition we had, namely

$$\operatorname{Re}(n^2/|n|^2) > 0$$

one has $\operatorname{Re}[f(z)] \gg 1$, although at the point $(0,0)$ we had $\operatorname{Re}[f(z)] = 0$. Therefore, after having left the first saddle point, the path of the integration must climb over a very high hill of $\operatorname{Re}[f(z)]$ in order to pass through the second saddle point. This climb causes the first saddle point to lose its significance and the second saddle point makes the value of I_3 arbitrarily large for arbitrarily large values of t_1 which was defined by Eq. (A2-21). However, a look at Figure (A2-3) shows that, the difference between the integrals $I_3(\theta = 0)$ and $I_3(\theta = \epsilon)$ where ϵ is arbitrarily small should be arbitrarily small and that for very large values of t_1 the main contribution to I_3 must still be due to the saddle point at the origin. Therefore, for very small values of θ the path of the integral must be kept away from the hills of the other saddle points and kept on the real axis. For the evaluation of I_3 a procedure similar to the one we had for $\theta = 0^0$ can be taken. This time instead of Eq. (A2-16) one has

$$f(z) = -(\alpha z^2 + \beta \sin^2 z) \exp(j2\Phi)$$

Similarly, Eq. (A2-17) changes into

$$f(z) \cong -r^2(\alpha + \beta) \exp(j2\Phi). \text{ Hence for } I_3 \text{ one has}$$

$$I_3 = \sqrt{\pi} \quad \omega_H / [\sqrt{8} k_0 n v_T \sqrt{\alpha + \beta}] \quad (\text{A2-27})$$

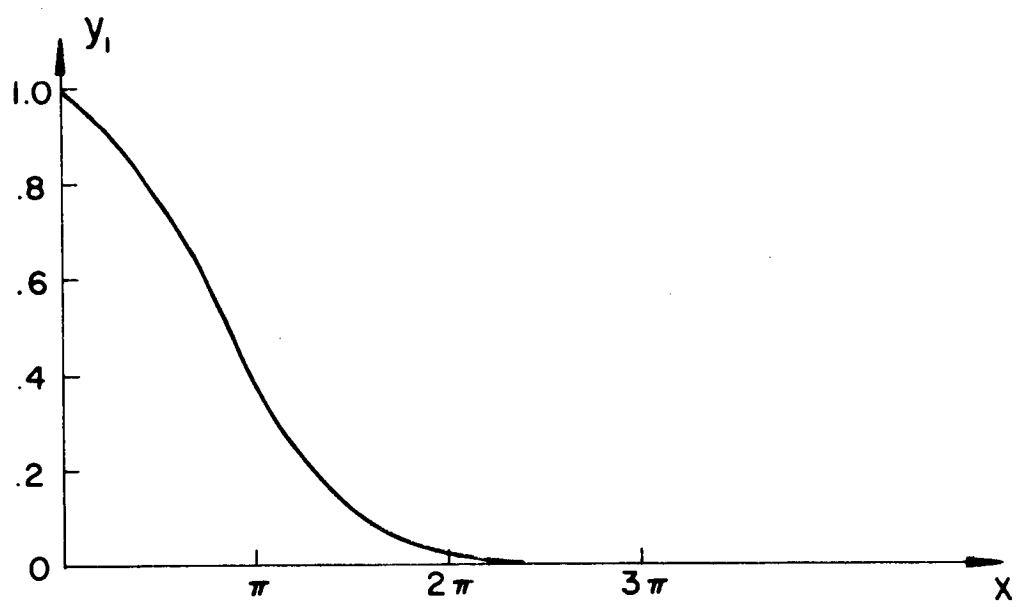


Figure (A2-3a) The plot of $y_1 = \exp(-t_1 x_1^2)$

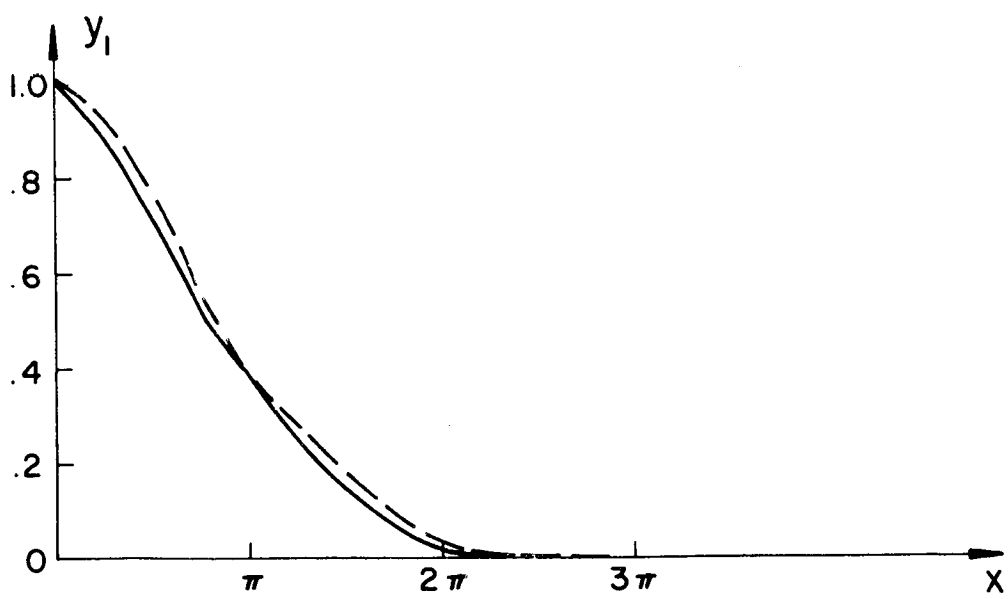


Figure (A2-3b) The plot of

$$y_1 = \exp[-t_1(\alpha x_1^2 + \beta \sin^2 x_1)] \text{ with } \alpha > 0, \beta > 0, \beta/\alpha \ll 1.$$

which again is independent of ν and hence one again has

$$I_1 = I_2 = I_3, \quad \theta \ll 1. \quad (\text{A2-28})$$

Using Eqs. (A2-27) and (A2-28) we find the matrix σ_r^1 as

$$(\sigma_r^1) = (s/n) \begin{bmatrix} 1 - (1/2)\sin^2\theta & -(1/2)\sin^2\theta & -[1/(2\sqrt{2})]\sin 2\theta \\ -(1/2)\sin 2\theta & 1 - (1/2)\sin^2\theta & -[1/(2\sqrt{2})]\sin 2\theta \\ -[1/(2\sqrt{2})]\sin 2\theta & -[1/(2\sqrt{2})]\sin 2\theta & \sin^2\theta \end{bmatrix} \quad (\text{A2-29})$$

For the case $\theta = 90^\circ$ the saddle points on the real axis have their abscissa as

$$z_p = P\pi/2 \quad P = 0, 1, 2,$$

Corresponding to these points one has

$$\left| \frac{d^2 f}{dz^2} \right|_{z=z_p} = \begin{cases} -n^2/|n|^2 & \text{for } P = 2m+1 \\ 0 & \text{for } P = 2m \end{cases} \quad (\text{A2-30})$$

Eq. (A2-30) shows that the main contribution to the integral comes from the saddle points at which p is an even integer. In the

vicinity of the P^{th} saddle point z_{op} , $f(z)$ will be

$$f(z) = -(n^2/|n|^2) [f(z_{\text{op}}) + (z - z_{\text{op}}) f'(z_{\text{op}}) + \dots]$$

inserting $n = |n| \exp(j\Phi)$

and $z - z_{\text{op}} = r \exp(j\psi)$

one finds for the minimum of $d_r[\text{Ref}(z)]$

$$\psi = P(\pi/2) - \Phi$$

Summing the contributions of all saddle points over the real axis one finds

$$I_3 = (1/2) (\pi/t)^{1/2} + \sum_{m=1} (\pi/t)^{1/2} \exp[-j(\omega - j\nu) 2m\pi/\omega_H] \\ - \sum_{m=0} (\pi/t)^{1/2} \exp[-t + j(\pi/2) - j(\omega - j\nu) (2m+1)\pi/\omega_H]$$

Assuming very little loss due to collisions which can be arbitrarily small, one can take

$$\nu \neq 0$$

in which case I_3 converges to

$$I_3 = \sqrt{\frac{\pi}{2}} \frac{\omega_H}{k_o n v_T} \left\{ \frac{1}{2} + \frac{1}{2j} \frac{\exp[-j(\omega - j\nu)\frac{\pi}{\omega_H}] \exp\left(\frac{-2k_o^2 n^2 v_T^2}{2} + j\frac{\pi}{2}\right)}{\sin[(\omega - j\nu)\frac{\pi}{\omega_H}]} \right\}$$

the term including $\exp(-2k_o^2 n^2 v_T^2 / \omega_H^2)$ is due to the saddle points at which $f(z)$ is other than zero and can be neglected since we assume that inequality (2-47) is valid.

Using Eqs. (A2-8.1) and (A2-8.2) one can evaluate I_1 and I_2 also.

Inserting the values found for the elements of the matrix (σ_r')

one finds

$$\sigma_{r11}' = \sigma_{r22}' = \frac{2s}{n} \left\{ \frac{3}{4} + \frac{\frac{3}{2} \exp \frac{-j(\omega - j\nu)\pi}{\omega_H} + j \left[\frac{3}{2} + \frac{2k_o^2 n^2 v_T^2}{2} \right] \exp \frac{k_o^2 n^2 v_T^2}{2}}{2j \sin \frac{(\omega - j\nu)\pi}{\omega_H}} \right\}$$

$$\sigma_{r12}' = \sigma_{r21}' = \frac{2s}{n} \left\{ -\frac{1}{4} + \frac{-\frac{1}{2} \exp \frac{-j(\omega - j\nu)\pi}{\omega_H} + j \left[\frac{1}{2} + \frac{2k_o^2 n^2 v_T^2}{2} \right] \exp \left[-\frac{2k_o^2 n^2 v_T^2}{2} \right]}{2j \sin \frac{(\omega - j\nu)\pi}{\omega_H}} \right\}$$

$$\sigma_{r13}' = \sigma_{r23}' = \sigma_{r31}' = \sigma_{r32}' = 0$$

$$\sigma_{r33}' = \frac{2s}{n} \left\{ \frac{1}{2} + \frac{\exp \frac{-j(\omega - j\nu)\pi}{\omega_H} - j \exp \frac{-k_o^2 n^2 v_T^2}{2}}{2j \sin \frac{(\omega - j\nu)\pi}{\omega_H}} \right\}$$

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